FORCING IDEALIZED

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A. NOTIONS OF DESCRIPTIVE SET THEORY IN GENERIC EXTENSIONS.

Let X be a Polish space, let $A \subset X$ be an analytic set, and let $V[G]$ be a generic extension of V.

Question. How do we interpret X and A in the extension?

Expected features. Interpretation of ω^ω is $(\omega^\omega)^{V[G]}$, interpretations of sets preserve usual set theoretic operations such as projection and countable union/intersection.

Interpretation of a complete metric space.

Definition. If $\langle X, d\rangle$ is a complete metric space in V, then its *interpretation* in $V[G]$ is a complete metric space $\langle Y, e\rangle$ in $V[G]$ together with a map $\phi : X \to Y$ which is an isometry of $\langle X, d\rangle$ with $\langle \mathrm{rng}(\phi), e\rangle$ and $\mathrm{rng}(\phi) \subset Y$ is dense.

Fact. An interpretation exists, as a completion of $\langle X, d\rangle$ and is unique up to a commuting diagram. The unique connecting map is an isometry.

Example. $\mathrm{id} : \mathbb{R}^V \to \mathbb{R}^{V[G]}$ is an interpretation of complete metric spaces.

Interpretation of a Polish space.

Definition. If $\langle X, \tau\rangle$ is a Polish space in V, then its interpretation in $V[G]$ is a map $\phi : X \to Y$ such that for some choices of complete metric d on X and e on Y, ϕ is an interpretation of $\langle X, d\rangle$ in $\langle Y, e\rangle$.

Fact. An interpretation exists, and is unique up to a commuting diagram. The unique map connecting two interpretations is a homeomorphism.

Example. id : $(\omega^\omega)^V \to (\omega^\omega)^{V[G]}$ is an interpretation of Polish spaces.

Interpretation of analytic sets.

Definition. If $\phi : X \to Y$ is an interpretation of a Polish space and $C \subset X$ is a closed set in V, then its interpretation C^{ϕ} is just the closure of $\phi''C$ in Y.

Definition. If $\phi : X \to Y$ is an interpretation and $A \subset X$ is an analytic set in V, then its interpretation $A^{\phi} \subset Y$ is $p(C^{\phi\times\psi})$, where $C \subset X \times X'$ is a closed set in V such that $A = p(C)$ and $\psi : X' \to Y'$ is an interpretation.

Fact. The interpretation of an analytic set is well-defined and it preserves usual operations such as projection or countable union or intersection. There is an obvious commutative diagram.

B. THE QUOTIENT FORCINGS.

Definition. Let I be a σ-ideal on a Polish space X. The symbol P_I denotes the poset of Borel I-positive sets ordered by inclusion.

- Which forcings belong to this class?
- Study them using the methods of descriptive set theory.
- Applications.

Combinatorial forcings and P_I.

Fact. (Sikorski) Every σ-algebra with countably many generators is of the form P_I.

Corollary. Every tree forcing is of the form P_I.

Counterexamples. Certain creature forcings are not of the form P_I even though the extension is given by a single real.

P_I adds a single real.

Theorem. If $G \subset P_I$ is a generic filter then there is a unique point $x_{\mathsf{gen}} \in X^{V[G]}$ such that for Borel $B \in P_I$, $B \in G$ if $x_{\mathsf{gen}} \in B^{V[G]}$. Thus, $V[G] = V[x_{\mathsf{gen}}]$.

Proof. Let $\bar{G}$ be the collection of all closed sets in G. $\bar{G}$ has the FIP and contains sets of arbitrarily small diameter, so $\bigcap \bar{G}$ is a singleton $\{x_{\mathsf{gen}}\}$. Now induct on Borel complexity of B.

Note. Thus, for every $B \in P_I$, $B \Vdash \dot{x}_{\mathsf{gen}} \in \dot{B}$.

Properness.

Definition. A poset P is proper if for every countable elementary submodel M of a large structure with $p \in M$, for every $p \in P \cap M$ there is $q \leq p$, a *master condition*, $q \Vdash \dot{G} \cap M$ is generic over M.

Theorem. P_I is proper if and only if for every M and every $B \in P_I \cap M$, the set $C = \{x \in X : x$ is P_I-generic over $M\}$ is I-positive.

Proof. The set C is Borel; $C = \bigcap_{D \in M} \bigcup (D \cap M)$. If $C \notin I$, then C is the master condition. If $C \notin I$ then there is no master condition.

C. EXAMPLES.

- For large classes of σ-ideals, the poset P_I is proper;
- there are natural improper examples as well as unresolved cases;
- no purely descriptive characterization of properness.

Theorem. If I is a σ-ideal σ-generated by closed sets then P_I is proper.

Proof. Let M be a countable model, $\{F_n : n \in \omega\}$ closed sets in I. To construct a point x generic over M, not in $\bigcup_n F_n$, by induction build $B_n \in P_I \cap M$ such that $B_0 \supset B_1 \supset \dots$, B_n in n-th dense set in M, and $B_n \cap F_n = 0$. Let x be the single point in $\bigcap_n B_n$.

Example. The Miller forcing.

Theorem. Let I be the σ-ideal on ω^ω σ-generated by compact sets. Then, the quotient P_I has a dense subset naturally isomorphic to Miller forcing.

Proof. Hurewicz theorem shows that every analytic set is either in I or it contains all branches of a superperfect tree. The map $T \mapsto [T]$ is then the isomorphism of Miller forcing with a dense subset of P_I.

Definition. Let X be Polish, $\mathcal{P}$ a countable collection of Borel sets, $w : \mathcal{P} \to \mathbb{R}^+$ a *weight* function. The *pavement submeasure* on X is defined by $\mu(B) = \inf\{\Sigma_n w(P_n) : B \subset \bigcup_n P_n\}$.

Theorem. Let μ be a pavement submeasure on X, let $I = \{B \subset X : \mu(B) = 0\}$. The poset P_I is proper.

Proof. Let $B \in P_I$, let M be a countable model with $B \in M$, let $\{P_n : n \in \omega\}$ be pavers with $\Sigma_n w(P_n) < \mu(B)$. To find a point x generic over M, $x \notin \bigcup_n P_n$, by induction on $m \in \omega$ build conditions B_m such that $B \supset B_0 \supset B_1 \supset \dots$, B_m in m-th dense set in M, and for some number n_m, $B_m \cap \bigcup_{n<n_m} P_n = 0$ and $\Sigma_{n \geq n_m} w(P_n) < \mu(B_m)$. Let x be the unique point in $\bigcap_m B_m$.

D. DESCRIPTIVE SET THEORY AND THE FORCING RELATION.

- Main goal: remove the forcing relation from arguments;
- use existing descriptive knowledge to identify new forcing properties;
- link combinatorial forcing proofs with descriptive ones.

Borel reading of names.

Theorem. If P_I is proper and $B \Vdash \dot{y} \in Y$ then there is a condition $C \subset B$ and a Borel function $f : C \to Y$ such that $C \Vdash \dot{y} = \dot{f}(\dot{x}_{\mathsf{gen}})$.

Proof. Let M be a countable submodel with $B, \dot{y} \in M$, let $C = \{x \in B : x$ is P_I-generic over $M\}$. For $x \in C$ let $f(x) = \dot{y}/x$. The function f works.

Theorem. If P_I is proper and $B \Vdash \dot{D} \subset Y$ is a Borel set then there is a condition $C \subset B$ and a Borel set $E \subset C \times Y$ such that $C \Vdash \dot{D} = \dot{E}_{\dot{x}_{\mathsf{gen}}}$.

The bounding property.

Definition. A forcing P is *bounding* if for every $x \in \omega^\omega$ in the extension there is $y \in \omega^\omega$ in the ground model such that $x < y$.

Theorem. Suppose that P_I is proper. Then P_I is bounding iff both of the following hold:

- every I-positive Borel set has a compact I-positive subset;
- the *continuous reading of names*: every Borel function on I-positive Borel domain is continuous on a Borel I-positive subset.

Note. If P_I is proper then any two Polish topologies on X giving the same Borel structure coincide on a Borel I-positive set.

Fubini properties.

Definition. Let I, J be σ-ideals on X, Y. I, J have the *Fubini property* if there are no Borel sets $B \subset X$, $C \subset Y$, and $D \subset B \times C$ such that $B \notin I$, $C \notin J$, vertical sections of D are in J and horizontal sections of $\neg D$ are in I.

Definition. A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ (or $\mathbf{\Delta}^1_2$ on $\mathbf{\Sigma}^1_1$, etc.) If for every analytic set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_y \in I\}$ is $\mathbf{\Pi}^1_1$ (or $\mathbf{\Delta}^1_2$ etc.)

Heuristic. Many ZFC theorems about quotient forcings of $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals. More complicated ideals often need large cardinals.

Fact. A quotient poset P_I of a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I, if proper, adds no dominating reals.

E. DETERMINED GAMES ON BOOLEAN ALGEBRAS.

- for a given forcing property of a poset P, find a two player game characterizing it;

- if $P = P_I$ for a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I(or more complicated with large cardinals), prove determinacy of the game via an unraveling argument;

- use the winning strategy to make strong conclusions.

Bounding game.

Definition. Let P be a poset. In game G, Player I plays maximal antichains $A_n \subset P$ and Player II plays finite subsets $B_n \subset A_n$. Player II wins if $\bigwedge_n \bigvee B_n \neq 0$.

Theorem. The poset P is bounding iff Player I has no winning strategy. If $P = P_I$ for $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I then the game is determined.

Application. (Fremlin) If P is c.c.c. and Player II has a winning strategy then the completion of P is a Maharam algebra. Thus, if I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, c.c.c. and bounding, P_I must be a Maharam algebra.

Baire category preservation.

Definition. Let P be a poset. In game G, Player I plays p_n, Player II responds with $q_n \leq p_n$. Player I wins if $\bigwedge_n \bigvee_{m>n} q_m \neq 0$.

Theorem. The poset P preserves Baire category iff Player II has no winning strategy. If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and P_I is proper then the game is determined.

Application. If P is c.c.c. and Player I has a winning strategy, then every real added by P is a Cohen real. Thus, if I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, c.c.c. and preserves Baire category, then P_I must be the Cohen forcing.

F. THE COUNTABLE SUPPORT ITERATION.

Let I be an *iterable* σ-ideal on a Polish space:

- it must have suitable interpretations in generic extensions;
- every I-positive analytic set must have an I-positive Borel subset;
- the quotient poset P_I must be proper;
- the latter two must hold in every forcing extension.

Then we can evaluate the σ-ideals associated with the countable support iteration of P_I.

Iterated Fubini power.

Definition. If I is a σ-ideal on a Polish space X and $\alpha \in \omega_1$ is an ordinal, let I^α be the σ-ideal generated by sets A_f, where $f : X^{<\alpha} \to I$ is an arbitrary function and $A_f = \{\vec{x} \in X^\alpha : \exists \beta \in \alpha\ \vec{x}(\beta) \in f(\vec{x} \restriction \beta)\}$.

Theorem. If I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ iterable σ-ideal and $\alpha \in \omega_1$ then $(P_I)^\alpha = P_{I^\alpha}$. Moreover, I^α is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

Fact. A similar theorem for the Laver forcing requires large cardinals or similar assumptions already for iteration of length 2.

G. THE COUNTABLE SUPPORT PRODUCT.

Let $\langle I_n : n \in \omega \rangle$ be σ-ideals on Polish spaces $\langle X_n : n \in \omega \rangle$.

- Is the poset $\prod_n P_{I_n}$ proper?

- What is the associated σ-ideal?

- What forcing properties are preserved under the product?

The rectangular Ramsey property.

Definition. The σ-ideals I, J have the *rectangular Ramsey property* if for Borel sets $B \subset X$, $C \subset Y$, and $D_n \subset B \times C$ such that $B \notin I$, $C \notin J$ and $B \times C = \bigcup_n D_n$, there are Borel sets $B' \subset B$ and $C' \subset C$ such that $B' \times C' \subset D_n$ for some fixed n.

Note. If I, J have the rectangular Ramsey property, then the collection of Borel subset of $X \times Y$ containing no rectangle $B \times C$ for Borel sets $B \notin I$ and $C \notin J$, is a σ-ideal, called the *box product* of I, J, denoted by $I \times J$.

Note. If this is the case then $P_I \times P_J$ is naturally isomorphic to a dense subset of $P_{I \times J}$.

A product preservation theorem.

Theorem. If each $I_n : n \in \omega$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal such that the quotient poset P_{I_n} is proper and bounding and preserves Baire category, then $\prod_n I_n$ has these properties again.

Proof. The argument depends on a determined Boolean game.

H. OPTIMALITY OF ITERATED MODELS.

Given an inequality $\mathfrak{x} < \mathfrak{y}$, we will prove

- If it holds in some forcing extension then it holds in a fixed c.s.i. extension;
- in this case, it also must hold in every extension satisfying a certain variation of Ciesielski-Pawlikowski Axiom.

The verification reduces to a statement about Borel sets.

Tame cardinal invariants.

Definition. $\mathfrak{x}$ is a *tame* cardinal invariant if it is defined as the minimal size of a set $A \subset 2^\omega$ such that $\phi(A) \wedge \psi(A)$, where

- universal quantifiers of ϕ range over 2^ω or A, existential quantifiers range over elements of 2^ω;

- $\psi(A) = \forall x \in 2^\omega \exists y \in A\ \theta(x, y)$ where θ does not mention A at all.

Example. $\mathtt{non}(J)$ is tame, if J is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal. $\mathfrak{a}$ is tame.

Example. $\mathfrak{h}$ is not a tame invariant.

Theorem. (LC) Whenever $\mathfrak{x}$ is a tame cardinal invariant and $\mathfrak{x} < \mathfrak{c}$ can be forced, then $\mathfrak{x} < \mathfrak{c}$ holds in the iterated Sacks model.

$\mathfrak{x}$ =smallest number of sets in an ideal necessary to cover the real line etc.

- All inequalities of the type $\mathfrak{x} < \mathfrak{c}$ are *mutually consistent*.

- All inequalities $\mathfrak{x} < \mathfrak{c}$ can be realized with $\aleph_1 = \mathfrak{x} < \mathfrak{c} = \aleph_2$.

- Elimination of forcing.

- Needs large cardinals for Woodin's Σ^2_1 absoluteness.

I. DUALITY THEOREMS

Theorem. If J is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal and ZFC proves $\mathtt{cov}(J) = \mathfrak{c}$ then $\mathtt{non}(J) \leq \aleph_2$.

Notation. $\mathtt{cov}(J)$ is the smallest size of a family of J-sets that covers the whole space; $\mathtt{non}(J)$ is the smallest size of a non-J set.

Other dualities possible. Exchanging $\mathtt{cov}, \mathtt{non}$ with $\mathtt{add}, \mathtt{cof}$. Exchanging $\mathfrak{c}, \aleph_2$ with $\mathfrak{hm}$ and $\aleph_{\omega_2+1}$.

The main point. $\texttt{non}(\texttt{ctble}^{\alpha}) \leq \aleph_2$ for every countable ordinal α.

Proof. Use ZFC club guessing on $\aleph_2$. Pcf theory gives the bound of $\aleph_{\omega+1}$.

Generalizations. Other dualities require a significantly sharper argument.

J. PRESERVATION THEOREMS.

Theorem. If I is an iterable $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal and J is a σ-ideal σ-generated by a coanalytic collection of closed sets, then Fubini property of I, J implies the Fubini property of I^α, J for every countable ordinal α.

Corollary. In this case, the countable support iteration of P_I preserves J-positive sets.

Example. Preservation of Baire category is preserved under the countable support iteration of P_I.

Definition. A σ-ideal I on Polish X has the *overspill property* if the closed countable sets cannot be separated from the I-positive closed sets by a Borel set.

Example. Countable sets, H-sets, ... vs. meager sets, null sets, sets of extended uniqueness etc.

Theorem. If I is iterable and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ with the overspill property then even I^α has the overspill property for all countable α.

Corollary. The countable support of P_I forces that the relevant space is covered by the ground model coded closed J-small sets, for every σ-ideal J without the overspill property.

Definition. A σ-ideal J on Polish X is *ergodic* if there is a countable Borel equivalence relation E such that every Borel J-invariant set is either J-small or its complement is J-small.

Theorem. If