Forcing with filters and ideals (part II.) Malykhin's Problem

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Filters and ideals

An ideal $\mathcal{I}$ on ω is

- *tall* if for every infinite $A \subseteq \omega$ there is an $I \in \mathcal{I}$ such that $|A \cap I|$ is infinite,
- *ω-hitting* if for every $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$ there is an $I \in \mathcal{I}$ such that $A_n \cap I$ is infinite for all $n \in \omega$,

Observation. If you split an ω-hitting ideal into countably many pieces, one of the pieces is ω-hitting.

- (Katětov order)Let $\mathcal{I}$ and $\mathcal{J}$. $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.
- (Katětov-Blass order) As above with a finite-to-one function f.

Mathias and Laver type forcings

Let $\mathcal{F}$ be a filter on ω. Then

$$\mathbb{M}_{\mathcal{F}} = \{(s, A) : s \in [\omega]^{<\omega} \text{ and } A \in \mathcal{F}\}$$

ordered by $(s, A) \leq (t, B)$ if $s \supseteq t$, $A \subseteq B$ and $s \setminus t \subseteq B$, and

$$\mathbb{L}_{\mathcal{F}} = \{T \subseteq \omega^{<\omega} : T \text{ is a tree with stem } s_T \text{ such that}$$
$$\text{for all } t \in T, t \supseteq s_T \Rightarrow succ_T(t) \in \mathcal{F}\},$$

ordered by inclusion.

$succ_T(t) = \{n \in \omega : t^\frown n \in T\}$,

Definition

Given $s \in \omega^{<\omega}$ and φ formula in the forcing language we say that s *favours* φ if no condition in $\mathbb{L}_{\mathcal{F}}$ with stem s forces "$\neg\varphi$".

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Preservation of ω-hitting

Definition

A forcing notion $\mathbb{P}$ *strongly preserves ω-hitting* if for every sequence $\langle \dot{A}_n \colon n \in \omega \rangle$ of $\mathbb{P}$-names for infinite subsets of ω there is a $\langle B_n \colon n \in \omega \rangle$ sequence of infinite subsets of ω such that for any $B \in [\omega]^\omega$, if $B \cap B_n$ is infinite for all n then $\Vdash_{\mathbb{P}}$ "$B \cap \dot{A}_n$ is infinite for all n".

Proposition (Brendle-H.)

Finite support iteration of forcings strongly preserving ω-hitting strongly preserves ω-hitting.

Preservation of ω-hitting by $\mathbb{L}_{\mathcal{F}}$

Back to $\mathbb{L}_{\mathcal{F}}$:

Lemma (Brendle-H.)

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F} = \mathcal{I}^*$ be the dual filter. Then the following are equivalent:

(1) For every $A \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction A$ the ideal $\mathcal{J}$ is not ω-hitting,

(2) $\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting, and

(3) $\mathbb{L}_{\mathcal{F}}$ preserves ω-hitting.

Proof: (2)⇒(3)⇒(1) is easy. To see (1) ⇒ (2), assume not, i.e. there is a sequence $\langle \dot{A}_n \colon n \in \omega \rangle$ of $\mathbb{P}$-names for infinite subsets of ω such that for any $\langle B_n \colon n \in \omega \rangle$ sequence of infinite subsets of ω there is a $B \in [\omega]^\omega$ such that $B \cap B_n$ is infinite for all n yet there is a condition $T_B \in \mathbb{L}_{\mathcal{F}}$ such that for some n_B, m_B $T_B \Vdash$ "$B \cap \dot{A}_{n_B} \subseteq m_B$".

Preservation of ω-hitting by $\mathbb{L}_{\mathcal{F}}$

... want to prove ...

For every $X \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction X$ the ideal $\mathcal{J}$ is not ω-hitting, $\Rightarrow$ $\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting.

Let $\mathcal{J} = \{B \in [\omega]^\omega : \exists T_B \in \mathbb{L}_{\mathcal{F}}, n_B, m_B \text{ s. t. } T_B \Vdash \text{“}B \cap \dot{A}_{n_B} \subseteq m_B\text{''}\}$.

Define

$rank_n(s) = 0$ iff either
(1) $\exists Z \subseteq \omega$ infinite $\forall k \in Z(s$ favours $k \in \dot{A}_n)$, or
(2) $\exists X \in \mathcal{I}^+$ and $f : X \to \omega$ $\forall l \in X$ $s^\frown l$ favours $f(l) \in \dot{A}_n$ and $\forall k \in \omega$ $f^{-1}(k) \in \mathcal{I}$

finally, $rank_n(s) \leq \alpha$ if $\{i : rank(s^\frown i) < \alpha\} \in \mathcal{I}^+$.

Claim: For all s, $rank_n(s) < \infty$.

(Hint: If not, construct a condition with stem s which forces $\dot{A}_n$ finite.)

Preservation of ω-hitting by $\mathbb{L}_{\mathcal{F}}$

... still want ...

For every $X \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction X$ the ideal $\mathcal{J}$ is not ω-hitting, $\Rightarrow$ $\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting.

Have $\mathcal{J} = \{B \in [\omega]^\omega : \ \exists T_B \in \mathbb{L}_{\mathcal{F}}, n_B, m_B \text{ s. t. } T_B \Vdash \text{“}B \cap \dot{A}_{n_B} \subseteq m_B\text{''}\}$ ω-hitting, and WLOG, for each B, $rank_{n_B}(s_{T_B}) = 0$.
Now, fix s, n such that $\mathcal{J}_0 = \{B \in \mathcal{J} : s_B = s \text{ and } n_B = n\}$ is ω-hitting.

Then either of the following leads to a contradiction:
Case 1. $\exists Z \subseteq \omega$ infinite $\forall k \in Z(s \text{ favours } k \in \dot{A}_n)$.
Case 2. $\exists X \in \mathcal{I}^+$ and $f : X \to \omega$ $\forall l \in X$ $s^\frown l$ favours $f(l) \in \dot{A}_n$ and $\forall k \in \omega$ $f^{-1}(k) \in \mathcal{I}$.

Preservation of ω-hitting by $\mathbb{L}_{\mathcal{F}}$

... keep wanting to prove:

For every $X \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction X$ the ideal $\mathcal{J}$ is not ω-hitting, $\Rightarrow$ $\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting.

Have s, n such that $\mathcal{J}_0 = \{B \in \mathcal{J} : s_B = s \text{ and } n_B = n\}$ is ω-hitting.

Case 1. $\exists Z \subseteq \omega$ infinite $\forall k \in Z (s \text{ favours } k \in \dot{A}_n)$.

Pick $B \in \mathcal{J}_0$ such that $B \cap Z$ is infinite and $k > m_B$ such that $k \in B \cap Z$. Then there is $S \leq T_B$ such that $S \Vdash$ "$k \in \dot{A}_n$", a contradiction.

Case 2. $\exists X \in \mathcal{I}^+$ and $f : X \to \omega$ $\forall l \in X$ $s^\frown l$ favours $f(l) \in \dot{A}_n$ and $\forall k \in \omega$ $f^{-1}(k) \in \mathcal{I}$.

$\mathcal{J}_0$ is ω-hitting, so there is a $B \in \mathcal{J}_0$ such that $f^{-1}[B] \in \mathcal{I}^+$. So there is a $k \in B \cap ran(f)$, $k > m_B$, such that $f^{-1}(k) \cap succ_{T_B}(s) \neq \emptyset$. Pick $j \in f^{-1}(k) \cap succ_{T_B}(s)$. Then $s^\frown j$ favours $k \in \dot{A}_n$ and hence there is a condition S whose stem extends $s^\frown j$ such that $S \Vdash$ "$k \in \dot{A}_n$".

Selected applications of $\mathbb{L}_\mathcal{F}$ and $\mathbb{M}_\mathcal{F}$

- Any inequality in the *Cichoń diagram* can be forced by a FSI of some combination of Random, $\mathbb{L}_\mathcal{F}$ and $\mathbb{M}_\mathcal{F}$ over a model of either *CH* or $MA + \neg CH$ (try it, it is fun).
- (Brendle) Consistency of $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ with large continuum.
- (Brendle) Consistently distinguish distributivity numbers of various σ-closed partial orders of size $\mathfrak{c}$.
- (Blass-Shelah, Brendle, Brendle-Fisher) Matrix iterations
- (H.- Ramos García) Consistency of every separable Fréchet group is metrizable.

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$\mathcal{I}^{<\omega}$ and ED topological groups

Recall that *fin* denotes the set of non-empty finite subsets of ω, and given $\mathcal{I}$ an ideal on ω

$$\mathcal{I}^{<\omega} = \{A \subseteq fin : (\exists I \in \mathcal{I})(\forall a \in A)\ a \cap I \neq \emptyset\}.$$

If $\mathcal{F} = \mathcal{I}^*$ then $(\mathcal{I}^{<\omega})^* = \mathcal{F}^{<\omega} = \langle [F]^{<\omega} : F \in \mathcal{F} \rangle$ induces a group topology $\tau_{\mathcal{I}}$ on the Boolean group $[\omega]^{<\omega}$ with the symmetric difference as the group operation by declaring $\mathcal{F}^{<\omega}$ the filter of neighbourhoods of the $\emptyset$.

Theorem (Louveau)

The group $([\omega]^{<\omega}, \tau_{\mathcal{I}})$ is extremally disconnected iff $\mathcal{F} = \mathcal{I}^*$ is a selective ultrafilter.

The same construction works on a measurable cardinal, and yet another example can be obtained from Matet forcing with a union-ultrafilter.

$\mathcal{I}^{<\omega}$ and ED topological groups

Question (Archangel'skii)

Is there a non-discrete extremally disconnected topological group?

Question

Let $\mathbb{G}$ be an extremally disconnected topological group and let $f : \mathbb{G} \to 2^\omega$ be a continuous function. Is there a non-empty open set $U \subseteq \mathbb{G}$ such that $f[U]$ is nowhere dense?

$\mathcal{I}^{<\omega}$ and Fréchet topological groups

Definition

A topological space X is *Fréchet* if for every $A \subseteq X$ and every $x \in \bar{A}$ there is a seqence $\langle x_n : n \in \omega \rangle$ of elements of A converging to x.

The topology $\tau_\mathcal{I}$ on $[\omega]^{<\omega}$ is Fréchet iff every $\mathcal{I}^{<\omega}$-positive set contains an infinite set in $(\mathcal{I}^{<\omega})^\perp$. Recall that if $\mathcal{I}$ is an ideal on a set X then

$$\mathcal{I}^\perp = \{J \subseteq X : (\forall I \in \mathcal{I})|I \cap J| < \omega\}.$$

$\tau_\mathcal{I}$ is metrizable if and only if the ideal $\mathcal{I}$ is countably generated.

Question (Reznichenko-Sipacheva, Gruenhage-Szeptycki)

Is there an uncountably generated $\mathcal{I}$ such that $\tau_\mathcal{I}$ is Fréchet?

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Malykhin's problem

Problem (Malykhin 1978)

Is every countable Fréchet group metrizable?

Partial negative solutions:

- $\mathfrak{p} > \omega_1$ . . . Yes
- (Gerlits-Nagy) There is an uncountable γ-set . . . Yes
- (Nyikos) $\mathfrak{p} = \mathfrak{b}$ . . . Yes
- (Ohrenstein-Tsaban) $\mathfrak{p} = \mathfrak{b}$ there is an uncountable γ-set.

Recall that a set of reals Y is a *γ-set* if every open ω-cover of Y has a γ-subcover. A cover $\mathcal{U}$ is an

- *ω-cover* if every finite subset of Y is contained in an element of $\mathcal{U}$,
- *γ-cover* if every element of Y is contained in all but finiely many elements of $\mathcal{U}$.

Malykhin's problem

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The solution

Theorem (H.-Ramos García)

It is consistent with **ZFC** that every separable Fréchet group is metrizable.

Plan of the proof:
Using a standard bookkeeping argument we construct a FS iteration of length ω_2 σ-centered forcing notions, eventually taking care of all countable Fréchet groups of weight less than ω_1. At stage α when dealing with the group $\mathbb{G}_\alpha$ handed to us by the bookkeeping device we need to do three things:

1. add a set $A \subseteq \mathbb{G}_\alpha$ which has the neutral element 0 as an accumulation point, and does not have a sequence converging to 0,
2. make sure that we do not add convergent sequences to the sets added earlier in the iteration.
3. make sure that 0 remains in the closure of A later on.

Fréchet idealized

- Given a space X and a point $x \in X$ we denote by $\mathcal{I}_x$ the dual ideal to the filter of neighbourhoods of x, $\mathcal{I}_x = \{A \subseteq X : x \notin \bar{A}\}$.
- If X is countable then the infinite members of $\mathcal{I}^\perp = \{J \subseteq X : (\forall I \in \mathcal{I}) |I \cap J| < \omega\}$ are exactly the sequences convergent to x.
- The space X is Fréchet at x iff every $\mathcal{I}_x$-positive set contains an infinite element of $\mathcal{I}_x^\perp$ iff $\mathcal{I}_x^{\perp\perp} = \mathcal{I}_x$ iff for no $A \in \mathcal{I}_x^+$ is the ideal $\mathcal{I}_x \upharpoonright A$ tall.

Definition

A forcing notion $\mathbb{P}$ *seals* an ideal $\mathcal{I}$ if it adds an $\mathcal{I}$-positive set A such that the ideal $\mathcal{I} \upharpoonright A$ is countably tall.

Sealing $\mathcal{I}$ by $\mathbb{L}_\mathcal{F}$

Lemma

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F}$ be al filter on ω.

IF

$\mathcal{I} \cap \mathcal{F} = \emptyset$ and for every countable family $\mathcal{H} \subseteq \mathcal{F}^+$ there is an $I \in \mathcal{I}$ such that $H \cap I \in \mathcal{F}^+$ for all $H \in \mathcal{H}$ (i.e. $\mathcal{I}$ is *ω-hitting w.r.t. $\mathcal{F}^+$*)

THEN

the forcing $\mathbb{L}_\mathcal{F}$ seals the ideal $\mathcal{I}$.

Proposition

Let $X = (\omega, \tau)$ be a regular Fréchet space, $x \in X$ be such that $\pi\chi(x, X) > \omega$. Let $\mathcal{G}$ be the filter of dense open subsets of X. Then:

(1) $\mathbb{L}_\mathcal{G}$ seals $\mathcal{I}_x$, and

(2) $\mathbb{L}_\mathcal{G}$ strongly preserves countable tallness.

Sealing $\mathcal{I}$ by $\mathbb{L}_{\mathcal{F}}$

... to be continued ...