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Forcing with filters and ideals (part I.)

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Contents

Filters and ideals

Definition

A family $\mathcal{I}$ of subsets of a (countable) set X is an *ideal* if it is (1) closed under subsets, (2) closed under finite unions, (3) $X \notin \mathcal{I}$ and (4) it contains all singletons of X. Dually, a family $\mathcal{F}$ of subsets of X is a *filter* if it is (1) closed under supersets, (2) closed under finite intersections (3) $\emptyset \notin \mathcal{F}$ and (4) it contains all co-finite subsets of X.

For an ideal $\mathcal{I}$ on X,

- $\mathcal{I}^* = \{X \setminus I : I \in \mathcal{I}\}$ is the *dual* filter (and the same for filters),
- $\mathcal{I}^+$ denotes $\mathcal{P}(X) \setminus \mathcal{I}$ (for filters $\mathcal{F}^+ = \mathcal{P}(X) \setminus \mathcal{F}^*$).

Special classes of filters and ideals

An ideal $\mathcal{I}$ on ω is

- *tall* if for every infinite $A \subseteq \omega$ there is an $I \in \mathcal{I}$ such that $|A \cap I|$ is infinite,
- a *P-ideal* if for every $\langle I_n : n \in \omega \rangle \subseteq \mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for all $n \in \omega$,
- *ω-hitting* if for every $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$ there is an $I \in \mathcal{I}$ such that $A_n \cap I$ is infinite for all $n \in \omega$,
- is a *P^+-ideal* if for every decreasing sequence $\langle X_n : n < \omega \rangle$ of $\mathcal{I}$-positive sets there is an $\mathcal{I}$-positive set X such that $X \subseteq^* X_n$, for all $n < \omega$.
- *meager, Borel, analytic,...* if it is meager, Borel, analytic,... as a subspace of $\mathcal{P}(\omega) \simeq 2^\omega$.

Every ω-hitting ideal is tall, and every tall P-ideal is ω-hitting.

Special ultrafilters

An ultrafilter $\mathcal{U}$ on ω is

- *selective* if for every partition $\{I_n : n \in \omega\}$ of ω into sets not in $\mathcal{U}$ there is $U \in \mathcal{U}$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.
- a *P-point* if for every partition $\{I_n : n \in \omega\}$ of ω into sets not in $\mathcal{U}$ there is $U \in \mathcal{U}$ such that $|U \cap I_n|$ is finite for every $n \in \omega$.
- a *Q-point* if for every partition $\{I_n : n \in \omega\}$ of ω into finite sets there is $U \in \mathcal{U}$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.
- *rapid* if the family of increasing enumerations of elements of $\mathcal{U}$ is dominating.
- *nowhere dense* (or a nwd-ultrafilter) if for every map $f : \omega \to \mathbb{R}$ there is a $U \in \mathcal{U}$ such that $f[U]$ is a nowhere dense subset of $\mathbb{R}$.

An ultrafilter $\mathcal{U}$ is selective iff it is both a P-point and a Q-point, every Q-point is rapid and every P-point is nwd.

Orderings on filters and ideals

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω.

- (Katětov order) $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.
- (Katětov-Blass order) $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$, for all $I \in \mathcal{I}$.
- (Rudin-Keisler order) $\mathcal{I} \leq_{RK} \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $A \in \mathcal{I}$ if and only if $f^{-1}[A] \in \mathcal{J}$.
- (Tukey order) $\mathcal{I} \leq_T \mathcal{J}$ if there is a function $f : \mathcal{I} \to \mathcal{J}$ such that for every $\subseteq$-bounded set $X \subseteq \mathcal{J}$, $f^{-1}[X]$ is $\subseteq$-bounded in $\mathcal{I}$.

Contents

Examples of forcing notions associated to filters/ideals

- Grigorieff forcing ... Silver restricted to a (non-meager P-)ideal
- Mathias-Prikry forcing ... Mathias forcing restricted to a filter
- Laver-Prikry forcing ... Laver forcing branching into a filter
- Matet-Prikry forcing ... Matet forcing restricted to a union-ultrafilter
- Sabok-Zapletal forcing ... Miller forcing branching into an $\mathcal{F}^+$ of a filter
- Farah-Zapletal forcings ... Mathias restricted and Laver branching to $\mathcal{F}^+$ of a filter
- Forcing $\mathcal{P}(\omega)/\mathcal{I}$ ... interesting for definable $\mathcal{I}$.
- Laflamme forcing ... ω^ω-bounding forcing associated to an F_σ-ideal
- P-ideal forcing of Zapletal ... natural forcing destroying a P-ideal
- Borel$(\mathcal{I})/\langle\mathcal{P}(I) : I \in \mathcal{I}\rangle$ ... natural forcing increasing the cofinality of a Borel ideal
- Forcing with classes of filters and/or ideals ... e.g. (Laflamme) for F_σ ideals.

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Destructibility of ideals by forcing

Definition

Given an ideal $\mathcal{I}$ and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *destroys* $\mathcal{I}$ if there is a $\mathbb{P}$-name $\dot{X}$ for an infinite subset of ω such that

$$\Vdash_{\mathbb{P}} \text{“} I \cap \dot{X} \text{ is finite for every } I \in \mathcal{I}\text{”}.$$

Destroying an ideal (which really means destroying *tallness* of the ideal) is, in the dual language of filters, called also *diagonalizing* or *zapping* a filter. The general question, central in combinatorial set theory of the reals, is the following:

Question

When does a given forcing destroy a given ideal?

Destructibility of ideals by forcing

Definition (Brendle)

Given a σ-ideal I on ω^ω, its *trace ideal* $tr(I)$ is an ideal on $\omega^{<\omega}$ defined by $A \in tr(I)$ if and only if $\{r : \exists^\infty n \in \omega\ (r|n \in A)\} \in I$.

Theorem (H.-Zapletal)

Let I be a σ-ideal on ω^ω. If P_I is a proper forcing with CRN then $\mathcal{P}(\omega^{<\omega})/tr(I)$ is a proper forcing as well and it is naturally isomorphic to a two-step iteration of P_I followed by an $\aleph_0$-distributive forcing.

Here P_I denotes the forcing consisting of I-positive Borel subsets of ω^ω, ordered by inclusion, where I is a σ-ideal on ω^ω, If P_I is a proper forcing then it has the *CRN* if for every Borel function $f : B \to 2^\omega$ with an I-positive Borel domain B there is an I-positive Borel set $C \subseteq B$ such that $f|C$ is continuous.

Destructibility of ideals by forcing

Theorem (H.-Zapletal)

Let P_I be a proper forcing with CRN, which is continuously homogeneous, and let $\mathcal{J}$ be an ideal on ω. Then the following conditions are equivalent:

(1) P_I *destroys* $\mathcal{J}$

(2) $\mathcal{J} \leq_K tr(I)$.

A forcing of the form P_I where I is a σ-ideal on ω^ω is *continuously homogeneous* if for every I-positive Borel set B there is a continuous function $F : \omega^\omega \to B$ such that $F^{-1}(A) \in I$ for all $A \in I|B$.

Observation

If $\mathcal{I} \leq_K \mathcal{J}$ and $\mathbb{P}$ destroys $\mathcal{J}$ then $\mathbb{P}$ destroys $\mathcal{I}$.

Destructibility of ideals by forcing

Theorem (Laflamme)

Every F_σ ideal can be destroyed by a proper ω^ω-bounding forcing.

Open problems

- (Roitman) Can every MAD family be destroyed by a proper ω^ω-bounding forcing?
- Can the density ideal $\mathcal{Z}$ be destroyed by a proper ω^ω-bounding forcing?
- Can every $F_{\sigma\delta}$ ideal (analytic P-ideal, or even just $\mathcal{Z}$) be destroyed by a proper forcing not adding a dominating real?
- Is there a Sacks-indestructible MAD family? (Yes, if $\mathfrak{b} = \mathfrak{a}$).
- (Steprāns) Is there a Cohen-indestructible MAD family?

Contents

Mathias and Laver type forcings

Recall that

$$\mathbb{M} = \{(s, A) : s \in [\omega]^{<\omega} \text{ and } A \in [\omega]^{\omega}\}$$

ordered by $(s, A) \leq (t, B)$ if $s \supseteq t$, $A \subseteq B$ and $s \setminus t \subseteq B$, and

$$\mathbb{L} = \{T \subseteq \omega^{<\omega} : T \text{ is a tree with stem } s_T \text{ such that}$$
$$\text{for all } t \in T, t \supseteq s_T \Rightarrow |succ_T(t)| = \omega\},$$

where $succ_T(t) = \{n \in \omega : t^\frown n \in T\}$, ordered by inclusion.
Given a family $\mathcal{X} \subseteq [\omega]^{\omega}$ call

$$\mathbb{M}_{\mathcal{X}} = \{(s, A) \in \mathbb{M} : A \in \mathcal{X}\}, \text{ and}$$

$$\mathbb{L}_{\mathfrak{X}} = \{T \in \mathbb{L} : \text{ for all } t \in T, t \supseteq s_T \Rightarrow succ_T(t) \in \mathcal{X}\}.$$

Theorem (Blass??)

If $\mathcal{F}$ is a selective ultrafilter then $\mathbb{M}_{\mathcal{F}} \simeq \mathbb{L}_{\mathcal{F}}$.

The separating number

If $\mathcal{F}$ is a filter on ω then $\mathbb{M}_{\mathcal{F}}$ and $\mathbb{L}_{\mathcal{F}}$ are σ-centered forcings. $\mathbb{M}_{\mathcal{F}}$ adds a generic subset $\dot{a}_{gen}$ of ω, while $\mathbb{L}_{\mathcal{F}}$ adds a generic function $\dot{f}_{gen}$ ($\in \omega^\omega$) and we let $\dot{a}_{gen}$ denote its range. Both forcings destroy (even *separate*) $\mathcal{J} = \mathcal{F}^*$:
$\dot{a}_{gen}$ is forced to be almost disjoint from all ground model sets in $\mathcal{J}$ and have an infinite intersection with all $\mathcal{J}$-positive ground model sets.

$$\begin{aligned}\mathrm{sep}(\mathcal{J}) = \min\{|\mathcal{H}| + |\mathcal{K}| : \mathcal{K} \subset \mathcal{J}, \mathcal{H} \subset \mathcal{J}^+ \text{ and } \\ \forall A \subset \omega\, ((\exists J \in \mathcal{K}(|A \cap J| = \omega) \text{ or } \exists H \in \mathcal{H}(|A \cap H| < \omega))\}.\end{aligned}$$

Proposition

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. If $\mathcal{I} \leq_{RK} \mathcal{J}$ then $\mathrm{sep}(\mathcal{J}) \leq \mathrm{sep}(\mathcal{I})$.

Martin's number for $\mathbb{L}_{\mathcal{F}}$

The σ-centered forcing $\mathbb{L}_{\mathcal{F}}$

- separates $\mathcal{F}^* = \mathcal{J}$,
- adds a dominating real, and
- (Błaszczyk-Shelah) adds a Cohen real iff $\mathcal{F}$ is not a nwd-ultrafilter.

Theorem (H.-Minami)

For every ideal $\mathcal{I}$ on ω

$$\mathfrak{m}(\mathbb{L}_{\mathcal{I}^*}) = \begin{cases} \min\{\mathfrak{b}, \mathrm{sep}(\mathcal{I})\} & \textit{if } \mathcal{I}^* \textit{ is nowhere dense ultrafilter, and} \\ \min\{\mathrm{add}(\mathcal{M}), \mathrm{sep}(\mathcal{I})\} & \textit{otherwise.} \end{cases}$$

Martin's number for $\mathbb{M}_{\mathcal{F}}$

Denote by *fin* the set of non-empty finite subsets of ω. Given $\mathcal{I}$ an ideal on ω, let

$$\mathcal{I}^{<\omega} = \{A \subseteq fin : (\exists I \in \mathcal{I})(\forall a \in A)\ a \cap I \neq \emptyset\}.$$

The σ-centered forcing $\mathbb{M}_{\mathcal{F}}$

- separates $(F^*)^{<\omega} = \mathcal{J}^{<\omega}$, (more precisely $\mathbb{M}_{\mathcal{F}} \times \mathbb{C}$ separates $\mathcal{J}^{<\omega}$), and
- (Blass ??, Mathias ??) adds a Cohen real iff $\mathcal{F}$ is not a selective ultrafilter.

Theorem (H.-Minami)

For every ideal $\mathcal{I}$ on ω

$$\mathfrak{m}(\mathbb{M}_{\mathcal{I}^*}) = \mathrm{sep}(\mathcal{I}^{<\omega}) \text{ if } \mathcal{I}^* \text{ is a selective ultrafilter, and}$$
$$\min\{\mathrm{cov}(\mathcal{M}), \mathrm{sep}(\mathcal{I}^{<\omega})\} \text{ otherwise.}$$

$\mathbb{M}_{\mathcal{F}}$ and dominating reals

When does $\mathbb{M}_{\mathcal{F}}$ add a dominating real?

- (Canjar) ($\mathfrak{d} = \mathfrak{c}$), There is an ultrafilter $\mathcal{U}$ such that the forcing $\mathbb{M}_{\mathcal{U}^*}$ does not add a dominating real (= *Canjar* ultrafilter).
- (Canjar) A Canjar ultrafilter is a P-point without rapid RK-predecessors.
- (Laflamme) Canjar $\Rightarrow$ strong P-point $\Rightarrow$ P-point without rapid RK-predecessors.

Definition

An ultrafilter $\mathcal{U}$ is a *strong P-point* if given a sequence $\langle C_n : n \in \omega \rangle$ of compact subsets of $\mathcal{U}$ there is a partition $\langle I_n : n \in \omega \rangle$ of ω into intervals such that whenever $U_n \in C_n$ for all $n \in \omega$ then

$$\bigcup_{n \in \omega} I_n \cap U_n \in \mathcal{U}.$$

$\mathbb{M}_{\mathcal{F}}$ and dominating reals

Theorem (H.-Minami)

Let $\mathcal{I}$ be an ideal on ω. Then $\mathbb{M}_{\mathcal{I}^*}$ does not add a dominating real if and only if the ideal $\mathcal{I}^{<\omega}$ is a P^+-ideal.

Theorem (Blass-H.-Verner)

Let $\mathcal{U}$ be an ultrafilter on ω. Then $\mathbb{M}_{\mathcal{I}^*}$ does not add a dominating real if and only if the ultrafilter $\mathcal{U}$ is a strong P-point.

Theorem (H.-Verner)

If $\mathcal{U}$ is $\mathcal{P}(\omega)/\mathcal{I}$-generic for an F_σ P-ideal then $\mathcal{U}$ is a P-point without rapid RK-predecessors which is not a strong P-point.

$\mathbb{M}_{\mathcal{F}}$ and dominating reals

Question (Brendle)

Is it consistent with ZFC that for every MAD family $\mathcal{A}$ the forcing $\mathbb{M}_{\mathcal{I}(\mathcal{A})^*}$ does not add a dominating real?

Theorem (H.-Martínez)

For every tall ideal $\mathcal{J}$ there is a MAD family $\mathcal{A}$ such that the forcing $\mathbb{M}_{\mathcal{I}(\mathcal{A})^*}$ destroys $\mathcal{J}$.

So,

- Brendle's question has a negative answer,
- There are no preservation theorems (other than general preservation theorems for σ-centered forcings) for forcings of the type $\mathbb{M}_{\mathcal{I}(\mathcal{A})^*}$.

$\mathbb{M}_{\mathcal{F}}$ and dominating reals

Theorem (Raghavan)

Shelah's forcing for increasing $\mathfrak{s}$ without increasing $\mathfrak{b}$ can be decomposed as a two step iteration $\mathbb{F} * \mathbb{M}_{\mathcal{U}}$, where $\mathbb{F}$ is the forcing with F_σ filters and $\mathcal{U}$ is the $\mathbb{F}$-generic ultrafilter.

Question

Let $\mathcal{I}$ be a Borel ideal. Is it true that $\mathbb{M}_{\mathcal{I}}$ does not add a dominating real if and only if $\mathcal{I}$ is F_σ?