Representations of ideals in Polish groups and in Banach spaces

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joint work (in progress) with

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Hejnice 2013

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Motivation: summable ideals

Definition

Let $h : \omega \to [0, \infty)$ be a sequence such that $\sum_{n\in\omega} h(n) = \infty$. Then the ***summable ideal associated to*** h is

$$\mathcal{I}_h = \left\{ A \subseteq \omega : \sum_{n\in A} h(n) < \infty \right\} \quad \text{(a } F_\sigma \text{ P-ideal).}$$

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Observation

We can also allow that $h : \omega \to \mathbb{R}$ IF we use ***unconditional (or u-)convergency***, that is, $A \in \mathcal{I}_h$ if the net

$$\sum h \restriction A = \left\{ s_h(F) = \sum_{n \in F} h(n) : F \in [A]^{<\omega} \right\} \text{ is convergent.}$$

Generalized summable ideals

Definition

Let G be an Abelian topological group and $h : \omega \to G$ such that $\sum_{n\in\omega} h(n)$ is not u-convergent. Then the ***generalized summable ideal associated to*** G ***and*** h is

$$\mathcal{I}_h^G = \mathrm{ideal}\left\{A \subseteq \omega : \sum h\restriction A \text{ is u-convergent}\right\}.$$

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Remarks

(1) $\{A \subseteq \omega : \sum h\restriction A$ is convergent$\}$ is not necessarily an ideal.

(2) If G is complete (i.e. Cauchy nets are convergent), then $\mathcal{I}_h^G = \{A \subseteq \omega : \sum h\restriction A$ is convergent$\}$.

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We say that an ideal $\mathcal{J}$ on ω is ***representable in G*** if there is an $h : \omega \to G$ such that $\mathcal{J} = \mathcal{I}_h^G$. If **C** is a class of groups then $\mathcal{J}$ is ***C-representable*** if it is representable in a $G \in$ **C**.

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(d) $\{G : \mathcal{J}$ is representable in $G\}$ for some fixed $\mathcal{J}$.

Examples

Example

The ***density zero ideal***

$$\mathcal{Z} = \left\{ A \subseteq \omega : \frac{|A \cap n|}{n} \to 0 \right\} = \left\{ A \subseteq \omega : \frac{|A \cap [2^n, 2^{n+1})|}{2^n} \to 0 \right\}$$

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is representable in c_0: Let $h(0) = 0$, and if $k \in [2^n, 2^{n+1})$ then let $h(k) = 2^{-n} e_k$ where $e_k = (\delta_{k,m})_{m \in \omega}$.
In other words. . .

Examples

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\begin{aligned}
h(0) &= (0, 0, \ 0 \ , \ 0 \ , \ 0 \ , \ldots) \\
h(1) &= (0, 1, \ 0 \ , \ 0 \ , \ 0 \ , \ldots) \\
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If $A \subseteq \omega$ then $\sum h \restriction A$ is u-convergent iff

$$\sum_{n \in A} h(n) = \left(0, \frac{|A \cap [2,4)|}{2}, \frac{|A \cap [4,8)|}{4}, \dots\right) \in c_0 \iff A \in \mathcal{Z}.$$

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Example

It is easy to see that if $(G_n)_{n\in\omega}$ is a sequence of discrete Abelian groups, then $\mathcal{J}$ is representable in $\prod_{n\in\omega} G_n$ iff there is a sequence $(X_n)_{n\in\omega}$ in $[\omega]^\omega$ such that

$$\mathcal{J} = \{A \subseteq \omega : \forall\, n \in \omega\ A \cap X_n \text{ is finite}\}.$$

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For example,

$$\{\emptyset\} \otimes \mathrm{Fin} = \{A \subseteq \omega \times \omega : \forall\, n \in \omega\ \{m : (n, m) \in A\} \text{ is finite}\}$$

has this property. It is a non tall $F_{\sigma\delta}$ P-ideal.

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Proof: (i) Let $X_{\mathcal{J}}$ be the linear subspace of ℓ_∞ generated by

$$\left\{ \sum_{n \in A} \frac{e_n}{n^2} : A \in \mathcal{J} \right\}$$

and let $h : \omega \to X_{\mathcal{J}}$, $h(n) = \frac{e_n}{n^2}$. Then $\mathcal{J} = \mathcal{I}_h^{X_{\mathcal{J}}}$.

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and let $h : \omega \to X_{\mathcal{J}}$, $h(n) = \frac{e_n}{n^2}$. Then $\mathcal{J} = \mathcal{I}_h^{X_{\mathcal{J}}}$.
(ii): Consider $\mathcal{J}$ as a subgroup of $(\mathcal{P}(\omega), \triangle)$, and let $h(n) = \{n\}$. Then $\mathcal{J} = \mathcal{I}_h^{\mathcal{J}}$.

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Corollary

There is a normed space X with $\dim(X) = 2^{\mathfrak{c}}$ such that all ideals on ω are representable in X.

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Corollary

There is a normed space X with $\dim(X) = 2^{\mathfrak{c}}$ such that all ideals on ω are representable in X.

Question (maybe easy)

Does there exist a normed space X such that all ideals on ω are representable in X and $\dim(X) < 2^{\mathfrak{c}}$? (No if $2^{\mathfrak{c}} = \mathfrak{c}^{+n}$ for some $n \in \omega$ because then $|X|^{\omega} = (\dim(X)^{<\omega}\mathfrak{c})^{\omega} = \dim(X)^{\omega} < 2^{\mathfrak{c}}$.)

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A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a ***submeasure*** (on ω) if

- $\varphi(\emptyset) = 0$;
- if $X, Y \subseteq \omega$ then $\varphi(X) \leq \varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$;
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φ is ***lower semicontinuous*** (lsc, in short) if $\varphi(X) = \lim_{n \to \infty} \varphi(X \cap n)$ for each $X \subseteq \omega$.

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If φ is an lsc submeasure then let

$$\mathrm{Exh}(\varphi) = \left\{ A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0 \right\}.$$

It is easy to see that if $\mathrm{Exh}(\varphi) \neq \mathcal{P}(\omega)$, then it is an $F_{\sigma\delta}$ P-ideal.

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Remark

The Polish topology on $\mathrm{Exh}(\varphi)$ is generated by the complete metric $d_\varphi(A, B) = \varphi(A \triangle B)$.

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Proof of "$\Leftarrow$": If φ is a lsc submeasure on ω, then

$$\mathrm{Exh}(\varphi) = \mathcal{I}_h^{\mathrm{Exh}(\varphi)} \quad \text{where } h : \omega \to \mathrm{Exh}(\varphi),\ h(n) = \{n\}.$$

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First proof of "$\Rightarrow$": Assume that $\mathcal{J} = \mathcal{I}_h^G$ for some Polish G and h.

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- The property "$\sum h \restriction A$ is u-convergent" is clearly $F_{\sigma\delta}$.
- Applying that we can fix a complete and *translation invariant* metric on G, it is easy to see that $\mathcal{I}_h^G$ is a P-ideal.

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An ideal $\mathcal{J}$ is Polish-representable iff $\mathcal{J}$ is an analytic P-ideal.

Second proof of "$\Rightarrow$": Let G, h, d be as above. Then

$$\varphi(A) = \sup \left\{ d\left(0, s_h(F)\right) : \emptyset \neq F \in [A]^{<\omega} \right\}$$

is a lsc submeasure and it is easy to check that $\mathcal{I}_h^G = \mathrm{Exh}(\varphi)$.

Banach-representability

A submeasure φ is ***non-pathological*** if for every $A \subseteq \omega$

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An analytic P-ideal $\mathcal{J}$ is ***non-pathological*** iff $\mathcal{J} = \mathrm{Exh}(\varphi)$ for some non-pathological lsc submeasure φ.
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$\mathcal{J}$ is non-pathological $\Rightarrow$ $\mathcal{J}$ is representable in ℓ_∞

Proof: Fix a sequence $(\mu_k)_{k\in\omega}$ of measures on ω such that

$$\varphi(F) = \sup\{\mu_k(F) : k \in \omega\} \quad \text{for every} \quad F \in [\omega]^{<\omega},$$

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and let $h : \omega \to \ell_\infty$ be defined by

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h(0) &= (\mu_0(\{0\}), \mu_1(\{0\}), \mu_2(\{0\}), \dots) \\
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Then $\|s_h(F)\| = \varphi(F)$ for every $F \in [\omega]^{<\omega}$ and so $\mathrm{Exh}(\varphi) = \mathcal{I}_h^{\ell_\infty}$.

$\mathcal{J}$ is Banach-representable $\Rightarrow$ $\mathcal{J}$ is non-pathological

Proof: Assume that $\mathrm{Exh}(\varphi) = \mathcal{I}_h^X$ for some Banach space X and $h : \omega \to X$. We will construct a non-pathological ψ such that $\mathrm{Exh}(\varphi) = \mathrm{Exh}(\psi)$.

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Let the submeasure $\widehat{\varphi}$ be defined by

$$\widehat{\varphi}(A) = \sup \left\{ \|s_h(A)\| : \emptyset \neq F \in [A]^{<\omega} \right\}.$$

We know that $\mathrm{Exh}(\widehat{\varphi}) = \mathcal{I}_h^X = \mathrm{Exh}(\varphi)$.

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- $a \in [\omega]^{<\omega} \rightsquigarrow a' \subseteq a$ such that $\widehat{\varphi}(a) = \|s_h(a')\|$.
- Fix an $x_a^* \in X^*$ with $\|x_a^*\| = 1$ and $x_a^*(s_h(a')) = \|s_h(a')\|$.

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We know that $\mathrm{Exh}(\widehat{\varphi}) = \mathcal{I}_h^X = \mathrm{Exh}(\varphi)$. How to construct ψ?

- $a \in [\omega]^{<\omega} \rightsquigarrow a' \subseteq a$ such that $\widehat{\varphi}(a) = \|s_h(a')\|$.
- Fix an $x_a^* \in X^*$ with $\|x_a^*\| = 1$ and $x_a^*\big(s_h(a')\big) = \|s_h(a')\|$.
- Let $\nu_a(b) = x_a^*\big(s_h(a' \cap b)\big)$, it is a signed measure.

$\mathcal{J}$ is Banach-representable $\Rightarrow$ $\mathcal{J}$ is non-pathological

Proof: Assume that $\mathrm{Exh}(\varphi) = \mathcal{I}_h^X$ for some Banach space X and $h : \omega \to X$. We will construct a non-pathological ψ such that $\mathrm{Exh}(\varphi) = \mathrm{Exh}(\psi)$.
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Then $\widehat{\varphi} \leq \psi \leq 2\widehat{\varphi}$ and so $\mathrm{Exh}(\psi) = \mathrm{Exh}(\widehat{\varphi}) = \mathrm{Exh}(\varphi)$.

My favorite question. . .

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Question

Which ideals are representable in c_0?

Thank you!