CONVERGENCE AND CHARACTER SPECTRA OF COMPACT SPACES

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Overview

- Basic definitions
- Hušek’s problem
- Inclusion in spectra
- Omission by spectra
- A problem on the $G_\delta$-topology
convergence spectrum

\[ A \rightarrow p \quad \text{if, for every neighbourhood } U \text{ of } p, \quad |A \setminus U| < |A| \]

\[ cS(p, X) = \{ |A| : A \subset X \text{ and } A \rightarrow p \} \]

is the convergence spectrum of \( p \) in \( X \)

\[ cS(X) = \bigcup \{ cS(x, X) : x \in X \} \]

is the convergence spectrum of \( X \)

\[ \chi(p, X) = \psi(p, X) = \kappa \geq \omega \Rightarrow \text{there is a 1-1 sequence } \langle x_\alpha : \alpha < \kappa \rangle \]

with \( x_\alpha \rightarrow p \); hence \( \kappa, \text{cf}(\kappa) \in cS(p, X) \)

In a compact \( T_2 \) space \( X \), \( \chi(p, X) = \psi(p, X) \) for all points \( p \in X \)
\( \chi S(p, X) = \{ \chi(p, Y) : p \text{ is non-isolated in } Y \subset X \} \)

is the character spectrum of \( p \) in \( X \)

\( \chi S(X) = \bigcup \{ \chi S(x, X) : x \in X \} \)

is the character spectrum of \( X \).

If \( X \) is compact \( T_2 \) then

\( \chi(p, Y) = \chi(p, \overline{Y}) \)

for any \( p \in Y \subset X \), so we may restrict to closed (i.e. compact) subspaces. This also implies:

For \( X \) compact \( T_2 \),

\( \chi S(p, X) \subset cS(p, X) \) and \( \kappa \in \chi S(p, X) \Rightarrow \text{cf}(\kappa) \in cS(p, X) \)
Hušek’s Problem

From here on, unless otherwise stated, space (usually denoted $X$) is compactum $\equiv$ infinite compact $T_2$ space

Note: $\omega \in cS(X) \iff \omega \in \chi S(X)$ and $\min cS(X) \leq \min \chi S(X) \leq 2^\omega$

Alexandrov-Urysohn (1920’s) : Is $\omega \in cS(X)$ ?

NO! Tychonov (1935), Čech, (1937) : $\omega /\in cS(\beta \omega)$

M. Hušek (1970’s) : Is $\min cS(X) \leq \omega_1$ ?

A. Dow (1989) : $V^{\mathbb{C}_\kappa} \models$ YES , if $V \models$ CH

I. J. (1993) : $V^{\mathbb{C}_{\omega_1}} \models \min \chi S(X) \leq \omega_1$ , for any $V$

I conjecture that ZFC $\vdash \min \chi S(X) \leq \omega_1$ , but don’t even know if

$$ZFC \vdash \chi S(X) \cap \text{REG} \neq \emptyset ?$$
**DEFINITION.**

\[ \{x_\alpha : \alpha < \varrho\} \text{ is free in } X \text{ if, for all } \alpha < \varrho, \]

\[ \{x_\beta : \beta < \alpha\} \cap \{x_\beta : \beta \geq \alpha\} = \emptyset \]

**THEOREM.** (J – Szentmiklóssy, 1991)

If there is a free sequence of length \( \varrho = \text{cf}(\varrho) > \omega \) in \( X \) then there is one converging to some \( p \in X \). Moreover, then

\[ \chi(p, \{x_\alpha : \alpha < \varrho\}) = \varrho. \]

Arhangel’skii: \( X \) is countably tight iff it has no uncountable free sequences. Hence Hušek’s problem is about countably tight compacta.

My original conjecture (true in \( V^{\mathbb{C}\omega_1} \)) : Any countably tight compactum has a point of character \( \leq \omega_1 \) (maybe isolated!).
Non-attributed results below are joint with W. Weiss

$$\hat{F}(X) = \min\{\kappa : \not\exists \text{ free sequence of length } \kappa \text{ in } X\}$$

**MAIN LEMMA.**

Let $X$ be a $T_3$ space with $\hat{F}(X) \leq \varrho \leq \text{cf}(\mu)$, moreover $p \in X$ with $\psi(p, X) \geq \mu$. Then either

(i) there is a discrete $D \in [X]^{<\varrho}$ with $p \in \overline{D}$ and $\psi(p, \overline{D}) \geq \mu$, or

(ii) there is a discrete $D \in [X]^{\varrho}$ such that $D \rightarrow p$.

$$\hat{t}(X) = \min \{\kappa : \forall A \subset X (\overline{A} = \cup \{\overline{B} : B \in [A]^{<\kappa}\})\}$$

Arhangel’skii: $\hat{t}(X) \leq \hat{F}(X) \leq \hat{t}(X)^+$ and if $\hat{t}(X)$ is regular then $\hat{t}(X) = \hat{F}(X)$. In particular, $X$ is countably tight iff

$$\hat{t}(X) = \hat{F}(X) = \omega_1$$
THEOREM 1.

If \( \chi(p, X) > \lambda = \lambda^{\dot{t}(X)} \) then \( \lambda \in \chi S(p, X) \). So, if \( X \) is countably tight and \( \chi(p, X) > \lambda = \lambda^\omega \) then \( \lambda \in \chi S(p, X) \).

COROLLARY. \( \chi(X) > c \) implies \( \omega_1 \in \chi S(X) \) or \( \{ c, c^+ \} \subset \chi S(X) \).

So, if \( \chi(X) > \omega \) then \( \chi S(X) \cap [\omega_1, c] \neq \emptyset \).

COROLLARY. If \( \kappa \) is strong limit and \( |X| \geq \kappa \) then

\[
\text{sup} \left( \kappa \cap \chi S(X) \right) = \kappa.
\]
inclusion theorems

NOTATION. \( dcS(p, X) = \{ |D| : D \subset X \text{ is discrete and } D \rightarrow p \} \)

\[ dcS(X) = \bigcup \{ dcS(x, X) : x \in X \} \]

**THEOREM 2.**

\[ \hat{F}(X) \leq \lambda = \text{cf}(\lambda) \text{ and } \chi(p, X) \geq \sum \{ (2^{\kappa})^+ : \kappa < \lambda \} \Rightarrow \lambda \in dcS(p, X). \]

**COROLLARY.** If \( \chi(X) > 2^{\kappa} \) then \( \kappa^+ \in dcS(X) \).

So, \( \chi(X) > c \Rightarrow \omega_1 \in dcS(X) \).
S omits $\kappa$ if $\kappa \notin S$ but there is a $\lambda \in S$ with $\lambda > \kappa$.

Tychonov (1935), Čech, (1937): $\omega \notin cS(\beta \omega) (\iff \omega \notin \chi S(\beta \omega))$; under CH, $\chi S(\beta \omega) = \{\omega_1\}$.

Fedorchuk (1977): $s = \omega_1$ implies $\exists X$ with $\chi S(X) = \{\omega_1\}$; if $2^{\omega_1} < \aleph_{\omega_1}$ then $cS(X) = \{\omega_1\}$ as well. But

$$\{\lambda < 2^{\omega_1} : \text{cf}(\lambda) = \omega_1\} \subset cS(X).$$

If $p > \omega_1$ then $\chi S(X) \neq \{\omega_1\}$ for all $X$. 
omitting uncountable cardinals 1.

The cardinality spectrum $S(X)$ of any top. space $Y$ is the set of cardinalities of all infinite closed subspaces of $Y$. 

**Lemma**

Let $Y$ be a locally compact $T_2$ space which is also locally $\mu$, and let $X = Y \cup \{p\}$ be the one-point compactification of $Y$. If $\mu < \kappa < |Y|$ and $\kappa \notin S(Y)$ then $\kappa \notin \chi S(X)$, while $|Y| = \chi(p, X)$.

**$\Phi(\kappa)$**

There are $T \in [\mathbb{R}]^\kappa$ and $A \subset [T]^\omega$ with $|A| = \kappa$ such that (i) for every $A \in A$ we have $|T \cap \overline{A}| = \kappa$ and (ii) for every $B \in [T]^\omega_1$ there is $A \in A$ with $A \subset B$.

**Theorem**

$\Phi(\kappa) \Rightarrow \exists$ locally countable and locally compact $T_2$ space $Y$ with $S(Y) = \{\omega, \kappa\}$, hence an $X$ with $\chi S(X) = \{\omega, \kappa\}$. 
Φ(\(c\)) is (trivially) true.

COROLLARY. (Hušek, 1981) \(\exists X \text{ s.t. } \chi S(X) = \{\omega, c\}\).

**Lemma**

If \(\kappa \leq \mathfrak{c}\) with \(\text{cf}(\kappa) \neq \omega_1\) and \(\langle [\kappa]^{\omega_1}, \subset \rangle\) has a dense subfamily of size \(\kappa\) then \(\Phi(\kappa)\) holds.

**Proposition**

Let \(\lambda\) be singular of countable cofinality s.t. \(\mu^{\omega_1} < \lambda\) whenever \(\mu < \lambda\). For every CCC partial order \(\mathbb{P}\) with \(|\mathbb{P}| = \lambda\), \(\langle [\lambda]^{\omega_1}, \subset \rangle\) has a dense subfamily of size \(\lambda\) in \(V^\mathbb{P}\). (A. Miller, for \(\mathbb{P} = C_\lambda\))

**Corollary**

If \(V \models GCH\) then, for any \(\kappa > \omega\), \(V^{C_\kappa} \models \Phi(\kappa)\).
omitting uncountable cardinals 3.

**Theorem**

Suppose $V \models GCH$ and $\lambda > \omega$ is a cardinal in $V$. Then, in $V^{C_\lambda}$, for every $\kappa \leq c$ there is a locally countable and locally compact $T_2$ space $Y$ with $S(Y) = \{\omega, \kappa\}$, hence there is a compactum $X$ with character spectrum $\chi S(X) = \{\omega, \kappa\}$.

Proof: $V^{C_\lambda} = (V^{C_\kappa})^{C_\lambda \setminus \kappa}$ and the properties of $Y$ are preserved.

**Corollary**

In $V^{C_\lambda}$, for every countable set $A$ of cardinals with $\omega \in A \subset [\omega, c]$ there is $X$ s.t. $\chi S(X) = A$.

**Theorem (L. Soukup)**

It is consistent with $c$ big that $\Phi(\kappa)$ holds for all $\kappa \leq c$. 
omitting uncountable cardinals 4.

Each example $X$ so far is the one-point compactification of a locally countable (loc. cpt) space, hence satisfies

$$cS(X) = [\omega, |X|].$$

**Theorem (J-Koszmider-Soukup, 2009)**

Consistently, there is $X$ s.t.

$$\chi S(X) = cS(X) = \{\omega, \omega_2\}.$$  

This is the **only** known example whose convergence spectrum is not convex on REG!
Any crowded $X$ has a crowded, hence non-discrete countable subspace.

If $\chi(p, X) > \omega$ for all $p \in X$, does $X_\delta$ have a non-discrete subspace of size $\omega_1$?

YES, if $\omega_1 \in cS(X)$, hence YES if $X$ is not countably tight.

YES for all $X$, if my old conjecture holds.