CONVERGENCE AND CHARACTER SPECTRA
OF COMPACT SPACES

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Overview

- Basic definitions
- Hušek’s problem
- Inclusion in spectra
- Omission by spectra
- A problem on the $G_\delta$-topology
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convergence spectrum

$A \to p$ if, for every neighbourhood $U$ of $p$, $|A \setminus U| < |A|$

$$cS(p, X) = \{ |A| : A \subset X \text{ and } A \to p \}$$

is the convergence spectrum of $p$ in $X$

$$cS(X) = \bigcup \{ cS(x, X) : x \in X \}$$

is the convergence spectrum of $X$

$$\chi(p, X) = \psi(p, X) = \kappa \geq \omega \Rightarrow \text{there is a 1-1 sequence } \langle x_\alpha : \alpha < \kappa \rangle \text{ with } x_\alpha \to p; \text{ hence } \kappa, \text{cf}(\kappa) \in cS(p, X)$$

In a compact $T_2$ space $X$, $\chi(p, X) = \psi(p, X)$ for all points $p \in X$
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If \( X \) is compact \( T_2 \) then

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From here on, unless otherwise stated, space (usually denoted $X$) is compactum $\equiv$ infinite compact $T_2$ space

Note: $\omega \in cS(X) \iff \omega \in \chi S(X)$ and $\min cS(X) \leq \min \chi S(X) \leq 2^\omega$

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I conjecture that $ZFC \vdash \min \chi S(X) \leq \omega_1$ , but don’t even know if

$ZFC \vdash \chi S(X) \cap \text{REG} \neq \emptyset$ ?
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free sequences

**DEFINITION.**

\( \{x_\alpha : \alpha < \varrho\} \) is free in \( X \) if, for all \( \alpha < \varrho \),

\[ \{x_\beta : \beta < \alpha\} \cap \{x_\beta : \beta \geq \alpha\} = \emptyset \]

**THEOREM.** (J – Szentmiklóssy, 1991)

If there is a free sequence of length \( \varrho = \text{cf}(\varrho) > \omega \) in \( X \) then there is one converging to some \( p \in X \). Moreover, then

\[ \chi(p, \{x_\alpha : \alpha < \varrho\}) = \varrho. \]

Arhangel’skii : \( X \) is countably tight iff it has no uncountable free sequences. Hence Hušek’s problem is about countably tight compacta.

My original conjecture (true in \( V^{\mathbb{C}\omega_1} \)) : Any countably tight compactum has a point of character \( \leq \omega_1 \) (maybe isolated!).
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main lemma for inclusion

Non-attributed results below are joint with W. Weiss

\[ \hat{F}(X) = \min\{\kappa : \not\exists \text{ free sequence of length } \kappa \text{ in } X\} \]

MAIN LEMMA.

Let \( X \) be a \( T_3 \) space with \( \hat{F}(X) \leq \varrho \leq \text{cf}(\mu) \), moreover \( p \in X \) with \( \psi(p, X) \geq \mu \). Then either

(i) there is a discrete \( D \in [X]^\varrho \) with \( p \in \overline{D} \) and \( \psi(p, \overline{D}) \geq \mu \), or

(ii) there is a discrete \( D \in [X]^\varrho \) such that \( D \rightarrow p \).

\[ \hat{t}(X) = \min \{\kappa : \forall A \subset X \ (\overline{A} = \bigcup \{\overline{B} : B \in [A]^{<\kappa}\})\} \]

Arhangel’skii: \( \hat{t}(X) \leq \hat{F}(X) \leq \hat{t}(X)^+ \) and if \( \hat{t}(X) \) is regular then \( \hat{t}(X) = \hat{F}(X) \). In particular, \( X \) is countably tight iff

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Non-attributed results below are joint with W. Weiss

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**MAIN LEMMA.**

Let \( X \) be a \( T_3 \) space with \( \hat{F}(X) \leq \rho \leq \text{cf}(\mu) \), moreover \( p \in X \) with \( \psi(p, X) \geq \mu \). Then either

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If $\chi(p, X) > \lambda = \lambda < \hat{t}(X)$ then $\lambda \in \chi S(p, X)$. So, if $X$ is countably tight and $\chi(p, X) > \lambda = \lambda^\omega$ then $\lambda \in \chi S(p, X)$.

COROLLARY. $\chi(X) > c$ implies $\omega_1 \in \chi S(X)$ or $\{c, c^+\} \subset \chi S(X)$. So, if $\chi(X) > \omega$ then $\chi S(X) \cap [\omega_1, c] \neq \emptyset$.

COROLLARY. If $\kappa$ is strong limit and $|X| \geq \kappa$ then 

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S omits $\kappa$ if $\kappa \notin S$ but there is a $\lambda \in S$ with $\lambda > \kappa$.

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omitting uncountable cardinals 1.

The cardinality spectrum $S(X)$ of any top. space $Y$ is the set of cardinalities of all infinite closed subspaces of $Y$.

Lemma

Let $Y$ be a locally compact $T_2$ space which is also locally $\mu$, and let $X = Y \cup \{p\}$ be the one-point compactification of $Y$. If $\mu < \kappa < |Y|$ and $\kappa \notin S(Y)$ then $\kappa \notin \chi S(X)$, while $|Y| = \chi(p, X)$.

$\Phi(\kappa)$

There are $T \in [\mathbb{R}]^\kappa$ and $A \subset [T]^{\omega}$ with $|A| = \kappa$ such that (i) for every $A \in A$ we have $|T \cap \overline{A}| = \kappa$ and (ii) for every $B \in [T]^{\omega_1}$ there is $A \in A$ with $A \subset B$.

Theorem

$\Phi(\kappa) \Rightarrow \exists$ locally countable and locally compact $T_2$ space $Y$ with $S(Y) = \{\omega, \kappa\}$, hence an $X$ with $\chi S(X) = \{\omega, \kappa\}$.
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There are $T \in [\mathbb{R}]^\kappa$ and $A \subseteq [T]^{\omega}$ with $|A| = \kappa$ such that (i) for every $A \in A$ we have $|T \cap \overline{A}| = \kappa$ and (ii) for every $B \in [T]^{\omega_1}$ there is $A \in A$ with $A \subseteq B$.

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