CONVERGENCE AND CHARACTER SPECTRA OF COMPACT SPACES

István Juhász
juhasz@renyi.hu

Alfréd Rényi Institute of Mathematics

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Overview

- Basic definitions
- Hušek's problem
- Inclusion in spectra
- Omission by spectra
- A problem on the G_δ-topology

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convergence spectrum

$A \to p$ if, for every neighbourhood U of p, $|A \setminus U| < |A|$

$$cS(p,X) = \{|A| : A \subset X \text{ and } A \to p\}$$

is the convergence spectrum of p in X

$$cS(X) = \cup\{cS(x,X) : x \in X\}$$

is the convergence spectrum of X

$\chi(p,X) = \psi(p,X) = \kappa \geq \omega \Rightarrow$ there is a 1-1 sequence $\langle x_\alpha : \alpha < \kappa \rangle$ with $x_\alpha \to p$; hence $\kappa, \mathrm{cf}(\kappa) \in cS(p,X)$

In a compact T_2 space X, $\chi(p,X) = \psi(p,X)$ for all points $p \in X$

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If X is compact T_2 then

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for any $p \in Y \subset X$, so we may restrict to closed (i.e. compact) subspaces. This also implies:

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A. Dow (1989) : $V^{\mathbb{C}_\kappa} \models$ YES , if $V \models$ CH

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I conjecture that ZFC $\vdash \min \chi S(X) \leq \omega_1$, but don't even know if

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free sequences

DEFINITION.

$\{x_\alpha : \alpha < \varrho\}$ is free in X if, for all $\alpha < \varrho$,

$$\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset$$

THEOREM. (J – Szentmiklóssy, 1991)

If there is a free sequence of length $\varrho = \mathrm{cf}(\varrho) > \omega$ in X then there is one converging to some $p \in X$. Moreover, then

$$\chi(p, \overline{\{x_\alpha : \alpha < \varrho\}} = \varrho\,.$$

Arhangel'skii : X is countably tight iff it has no uncountable free sequences. Hence Hušek's problem is about countably tight compacta.

My original conjecture (true in $V^{\mathbb{C}_{\omega_1}}$) : Any countably tight compactum has a point of character $\leq \omega_1$ (maybe isolated!).

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main lemma for inclusion

Non-attributed results below are joint with W. Weiss

$$\widehat{F}(X) = \min\{\kappa : \neg\exists \text{ free sequence of length } \kappa \text{ in } X\}$$

MAIN LEMMA.

Let X be a T_3 space with $\widehat{F}(X) \le \varrho \le \mathrm{cf}(\mu)$, moreover $p \in X$ with $\psi(p, X) \ge \mu$. Then either
(i) there is a discrete $D \in [X]^{<\varrho}$ with $p \in \overline{D}$ and $\psi(p, \overline{D}) \ge \mu$, or
(ii) there is a discrete $D \in [X]^{\varrho}$ such that $D \to p$.

$$\widehat{t}(X) = \min\left\{\kappa : \forall A \subset X \left(\overline{A} = \cup\{\overline{B} : B \in [A]^{<\kappa}\}\right)\right\}$$

Arhangel'skii : $\widehat{t}(X) \le \widehat{F}(X) \le \widehat{t}(X)^+$ and if $\widehat{t}(X)$ is regular then $\widehat{t}(X) = \widehat{F}(X)$. In particular, X is countably tight iff

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Non-attributed results below are joint with W. Weiss

$$\widehat{F}(X) = \min\{\kappa : \neg\exists \text{ free sequence of length } \kappa \text{ in } X\}$$

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Let X be a T_3 space with $\widehat{F}(X) \le \varrho \le \mathrm{cf}(\mu)$, moreover $p \in X$ with $\psi(p, X) \ge \mu$. Then either
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If $\chi(p,X) > \lambda = \lambda^{<\hat{t}(X)}$ then $\lambda \in \chi S(p,X)$. So, if X is countably tight and $\chi(p,X) > \lambda = \lambda^{\omega}$ then $\lambda \in \chi S(p,X)$.

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$$dcS(X) = \cup\{dcS(x, X) : x \in X\}$$

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$\widehat{F}(X) \le \lambda = \mathrm{cf}(\lambda)$ and $\chi(p, X) \ge \sum\{(2^\kappa)^+ : \kappa < \lambda\} \Rightarrow \lambda \in dcS(p, X)$.

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S omits κ if $\kappa \notin S$ but there is a $\lambda \in S$ with $\lambda > \kappa$.

Tychonov (1935), Čech, (1937) : $\omega \notin cS(\beta\omega)(\Leftrightarrow \omega \notin \chi S(\beta\omega))$;
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The cardinality spectrum $S(X)$ of any top. space Y is the set of cardinalities of all infinite closed subspaces of Y.

Lemma

Let Y be a locally compact T_2 space which is also locally μ, and let $X = Y \cup \{p\}$ be the one-point compactification of Y. If $\mu < \kappa < |Y|$ and $\kappa \notin S(Y)$ then $\kappa \notin \chi S(X)$, while $|Y| = \chi(p, X)$.

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There are $T \in [\mathbb{R}]^\kappa$ and $\mathcal{A} \subset [T]^\omega$ with $|\mathcal{A}| = \kappa$ such that (i) for every $A \in \mathcal{A}$ we have $|T \cap \overline{A}| = \kappa$ and (ii) for every $B \in [T]^{\omega_1}$ there is $A \in \mathcal{A}$ with $A \subset B$.

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$\Phi(\kappa) \Rightarrow \exists$ locally countable and locally compact T_2 space Y with $S(Y) = \{\omega, \kappa\}$, hence an X with $\chi S(X) = \{\omega, \kappa\}$.

omitting uncountable cardinals 1.

The cardinality spectrum $S(X)$ of any top. space Y is the set of cardinalities of all infinite closed subspaces of Y.

Lemma

Let Y be a locally compact T_2 space which is also locally μ, and let $X = Y \cup \{p\}$ be the one-point compactification of Y. If $\mu < \kappa < |Y|$ and $\kappa \notin S(Y)$ then $\kappa \notin \chi S(X)$, while $|Y| = \chi(p, X)$.

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$\Phi(\mathbf{c})$ is (trivially) true.

COROLLARY. (Hušek, 1981) $\exists X$ s.t. $\chi S(X) = \{\omega, \mathbf{c}\}$.

Lemma

If $\kappa \leq \mathbf{c}$ with $\mathrm{cf}(\kappa) \neq \omega_1$ and $\langle [\kappa]^{\omega_1}, \subset \rangle$ has a dense subfamily of size κ then $\Phi(\kappa)$ holds.

Proposition

Let λ be singular of countable cofinality s.t. $\mu^{\omega_1} < \lambda$ whenever $\mu < \lambda$. For every CCC partial order $\mathbb{P}$ with $|\mathbb{P}| = \lambda$, $\langle [\lambda]^{\omega_1}, \subset \rangle$ has a dense subfamily of size λ in $V^{\mathbb{P}}$. (A. Miller, for $\mathbb{P} = \mathbb{C}_\lambda$)

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If $V \models GCH$ then, for any $\kappa > \omega$, $V^{\mathbb{C}_\kappa} \models \Phi(\kappa)$.

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Proof: $V^{\mathbb{C}_\lambda} = \left(V^{\mathbb{C}_\kappa}\right)^{\mathbb{C}_{\lambda \setminus \kappa}}$ and the properties of Y are preserved.

Corollary

In $V^{\mathbb{C}_\lambda}$, for every countable set A of cardinals with $\omega \in A \subset [\omega, \mathbf{c}]$ there is X s.t. $\chi S(X) = A$.

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It is consistent with $\mathbf{c}$ big that $\Phi(\kappa)$ holds for all $\kappa \le \mathbf{c}$.

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Theorem (J-Koszmider-Soukup, 2009)

Consistently, there is X s.t.

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G_δ – topology

FACT.

Any crowded X has a crowded, hence non-discrete countable subspace.

PROBLEM.

If $\chi(p, X) > \omega$ for all $p \in X$, does X_δ have a non-discrete subspace of size ω_1?

YES, if $\omega_1 \in cS(X)$, hence YES if X is not countably tight.

YES for all X, if my old conjecture holds.

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