Submeasures and signed measures

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Winter School

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Maharam’s problem (1947):

- Is every Maharam algebra a measure algebra?
- Does every exhaustive additive vector measure admit a control measure?
- Does every exhaustive submeasure fail to be pathological?
- Is every exhaustive submeasure uniformly exhaustive?
- Is every exhaustive submeasure equivalent to a measure?
- In 2006 M. Talagrand constructed a ZFC counter example to Maharam’s problem.
- This solution is very uncooperative!
- Does the corresponding non-measurable Maharam algebra contain the random algebra as a complete subalgebra (i.e. does it add a random real)?
- Can we eliminate AC from the construction (i.e. eliminate the use of an ultrafilter)?
- Is this complete Boolean algebra homogenous?
- Can we generalise this construction to clopen $(2^\kappa)$?
- What else can we say about the relationship between submeasures and measures (keeping the Maharam problem in mind)?
- I will discuss a linear association between the collection of all submeasures on the clopen sets of the Cantor space and the space of signed measures on this algebra.
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Some definitions

Throughout $\mathcal{B}$ will always denote a Boolean algebra.

A map $\lambda: \mathcal{B} \rightarrow \mathbb{R}$ is called a signed measure if, for every disjoint $a$ and $b$ from $\mathcal{B}$, we have $\lambda(a \cup b) = \lambda(a) + \lambda(b)$; a measure if it is a signed measure but only assumes non-negative values from $\mathbb{R}$; a submeasure if the following conditions hold:

1. $\lambda(0) = 0$,
2. $\lambda(a) \leq \lambda(b)$, for every $a$ and $b$ such that $a \leq b$,
3. $\lambda(a \cup b) \leq \lambda(a) + \lambda(b)$, always.

These are all examples of functionals, which is to say that each satisfies $\lambda(0) = 0$. 
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Two functionals $\mu$ and $\lambda$ on $B$ are called equivalent if, for every sequence $(a_n)$ from $B$, we have

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A functional $\lambda$ on $B$ is called exhaustive if, for every antichain $(a_n)$ from $B$, we have

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Definition
Call a collection \( \{ a_i : i \in [n] \} \subseteq B, \) \(*\)-free if for every non-empty \( J \subseteq [n] \) we have
\[
\bigcap_{j \in J} a_j \cap \bigcap_{j \not\in J} a_{c_j} \neq 0
\]
\( \land \)
\[
\bigcup_{i \in [n]} a_i = 1.
\]

Remark: Recall that the collection \( \{ a_i : i \in [n] \} \) is free if for every \( J \subseteq [n] \) we have
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in which case, by considering \( J = \emptyset \), we would have
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Definition

Call a collection \( \{ a_i : i \in [n] \} \subseteq \mathcal{B} \) \textbf{*-free} if for every non-empty \( J \subseteq [n] \) we have

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Theorem
For every countable Boolean algebra $\mathcal{A}$ there exists a countable Boolean algebra $\mathcal{B}$ and an injective map $f: \mathcal{A} \to \mathcal{B}$ with the following properties:

1. $\mathcal{B} = \langle f[\mathcal{A}] \rangle$;
2. If $\mathcal{A}' \subseteq \mathcal{A}$ is a finite subalgebra, then the collection $f[\text{atoms}(\mathcal{A}')]$ is $\ast$-free in $\mathcal{B}$;
3. $\forall a, b \in \mathcal{A}$, $(f(a \cup b) = f(a) \cup f(b))$.

Moreover, if $D$ is a Boolean algebra and $g: \mathcal{A} \to D$ satisfies the above, then for any functional $\mu$ on $\mathcal{A}$, there exists a unique signed measure $\lambda$ on $D$ such that $\mu(a) = \lambda(g(a))$, for each $a \in \mathcal{A}$. 
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For every countable Boolean algebra \( A \) there exists a countable Boolean algebra \( B \) and an injective map \( f : A \to B \) with the following properties:

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(T.3) \( (\forall a, b \in A)(f(a \cup b) = f(a) \cup f(b)) \).
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Moreover, if $\mathcal{D}$ is a Boolean algebra and $g : \mathcal{A} \to \mathcal{D}$ satisfies the above, then for any functional $\mu$ on $\mathcal{A}$, there exists a unique signed measure $\lambda$ on $\mathcal{D}$ such that $\mu(a) = \lambda(g(a))$, for each $a \in \mathcal{A}$. 
The basic idea

Let $A$ be the finite Boolean algebra of two atoms $a$ and $b$ and define the functional $\mu: A \to \mathbb{R}$ by:

$\mu(a) = \frac{3}{4}$
$\mu(b) = \frac{3}{4}$

This is not additive, since $a$ and $b$ cannot assume these values and be disjoint at the same time ($\frac{3}{4} + \frac{3}{4} \neq 1$).

If we want it to be additive and maintain these values, we will need $a$ and $b$ to intersect.

So we arrive at the Boolean algebra $B$ of three atoms $c, d, e$ and the measure $\lambda: B \to \mathbb{R}$ defined by:

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- So we arrive at the Boolean algebra \( \mathcal{B} \) of three atoms \( c, d \) and \( e \) and the measure \( \lambda : \mathcal{B} \rightarrow \mathbb{R} \) defined by

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\lambda(c) = \lambda(e) = \frac{1}{4} \quad \text{and} \quad \lambda(d) = \frac{1}{2}.
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The basic idea

We are in fact solving the following system of linear equations:

\[ \mu(a) = \frac{3}{4} \]
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\[ \lambda(c) = \frac{1}{4} \]
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By constructing an appropriate matrix and showing that it is invertible, we see that in general this can be done for any finite Boolean algebra. The final construction is obtained as a direct limit of these finite constructions.
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We are in fact solving the following system of linear equations:

1. \( \mu(a \cup b) = \lambda(c) + \lambda(d) + \lambda(e); \)
2. \( \mu(a) = \lambda(c) + \lambda(d); \)
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We can construct this map (almost) explicitly.
Explicit construction

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- Let $X_1, X_2, \ldots$ be a sequence of finite non-empty sets, let $X^{(n)} = \prod_{i \in [n]} X_i$ and $X = \prod_{i \in \mathbb{N}} X_i$. 

- Define another sequence of finite non-empty sets $T_1, T_2, \ldots$ as follows.

- Let $T_1 = \mathcal{P}(X_1)$ and $T_{i+1} = \{A \subseteq X^{(i+1)} : \text{every member of } X^{(i)} \text{ has an extension in } A\}$. 

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- Our $B$ will be the clopen sets of $T := \prod_{i \in \mathbb{N}} T_i$ and let $T^{(n)} = \prod_{i \in [n]} T_i$.

- Say that $s \in X^{(n)}$ generates $t \in T^{(n)}$ if $(\forall i \in [n])(s \upharpoonright [i] \in t(i))$.

- Now define $f$ by $f([s]) = \bigcup \{[t] : s \text{ generates } t\}$. 

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- Let $T_1 = \mathcal{P}(X_1)^+$ and
  \[ T_{i+1} = \{ A \subseteq X^{(i+1)} : \text{every member of } X^{(i)} \text{ has an extension in } A \}. \]
- Our $\mathcal{B}$ will be the clopen sets of $T := \prod_{i \in \mathbb{N}} T_i$ and let $T^{(n)} = \prod_{i \in [n]} T_i$.
- Say that $s \in X^{(n)}$ generates $t \in T^{(n)}$ if
  \[ (\forall i \in [n])(s \upharpoonright [i] \in t(i)). \]
- Now define $f$ by
  \[ f([s]) = \bigcup \{ [t] : s \text{ generates } t \}. \]
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Warning! If each $X_i = \{1, 2\}$ then $\mu$ is not exhaustive.
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**Lemma**

If $\mu$ is a submeasure and the corresponding signed measure is non-negative, then $\mu$ must dominate a non-trivial measure (i.e. it cannot be pathological).
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*If $\mu$ is a submeasure and the corresponding signed measure is non-negative, then $\mu$ must dominate a non-trivial measure (i.e. it cannot be pathological).*

In particular the signed measure corresponding to Talagrand’s submeasure is indeed non-negative.
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**Lemma**

*If* $\mu$ *is a submeasure and the corresponding signed measure is non-negative, then $\mu$ must dominate a non-trivial measure (i.e. it cannot be pathological).*

In particular the signed measure corresponding to Talagrand's submeasure is indeed non-negative.

On the other hand, there are very simple submeasures where the corresponding signed measure is unbounded. For example take the submeasure

$$
\mu(a) = \begin{cases} 
1, & \text{if } a = 1; \\
\frac{1}{2}, & \text{otherwise.}
\end{cases}
$$
The End