Submeasures and signed measures

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  - Can we generalise this construction to $\mathrm{clopen}(2^\kappa)$?
- What else can we say about the relationship between submeasures and measures (keeping the Maharam problem in mind)?
- I will discuss a linear association between the collection of all submeasures on the clopen sets of the Cantor space and the space of signed measures on this algebra.

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- **submeasure** if the following conditions hold:
  - $\lambda(0) = 0$,
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These are all examples of **functionals**, which is to say that each satisfies $\lambda(0) = 0$.

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Maharam's problem: *Is every exhaustive submeasure on the clopen sets of the Cantor space equivalent to a measure*?

Submeasures and signed measures

Definition

Call a collection $\{a_i : i \in [n]\} \subseteq \mathfrak{B}$, $*$-**free** *if for every non-empty* $J \subseteq [n]$ *we have*

$$\left(\bigcap_{j\in J} a_j\right) \cap \left(\bigcap_{j\notin J} a_j^c\right) \neq 0 \wedge \bigcup_{i\in[n]} a_i = 1.$$

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Remark: Recall that the collection $\{a_i : i \in [n]\}$ is **free** if *for every* $J \subseteq [n]$ we have

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in which case, by considering $J = \emptyset$, we would have $\bigcup_{i \in [n]} a_i \neq 1$.

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For every countable Boolean algebra $\mathfrak{A}$ *there exists a countable Boolean algebra* $\mathfrak{B}$ *and an injective map* $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ *with the following properties:*

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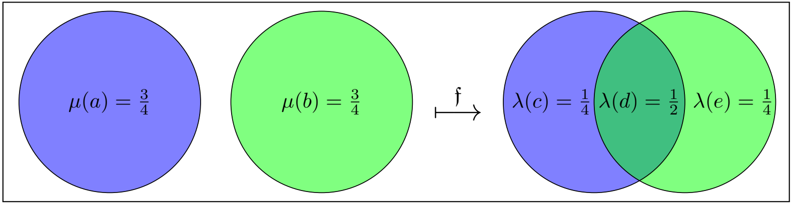
Moreover, if $\mathfrak{D}$ is a Boolean algebra and $\mathfrak{g}:\mathfrak{A}\to\mathfrak{D}$ satisfies the above, then for any functional μ on $\mathfrak{A}$, there exists a unique signed measure λ on $\mathfrak{D}$ such that $\mu(a)=\lambda(\mathfrak{g}(a))$, for each $a\in\mathfrak{A}$.

The basic idea



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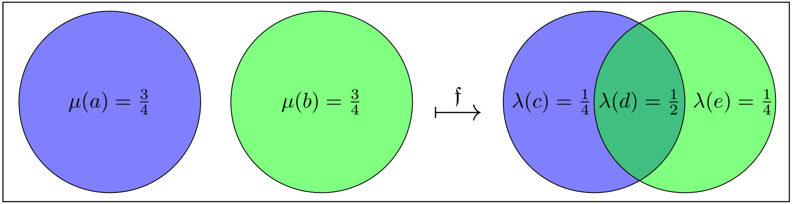

Let $\mathfrak{A}$ be the finite Boolean algebra of two atoms a and b and define the functional $\mu : \mathfrak{A} \to \mathbb{R}$ by:

$$\mu(a) = \mu(b) = \frac{3}{4} \text{ and } \mu(a \cup b) = 1.$$

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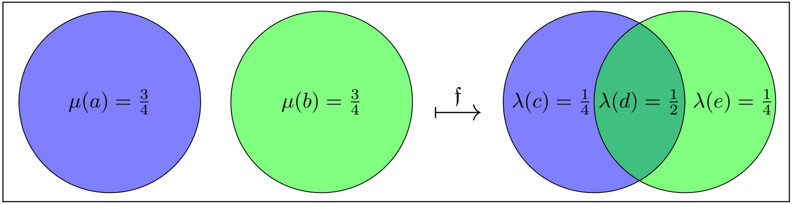

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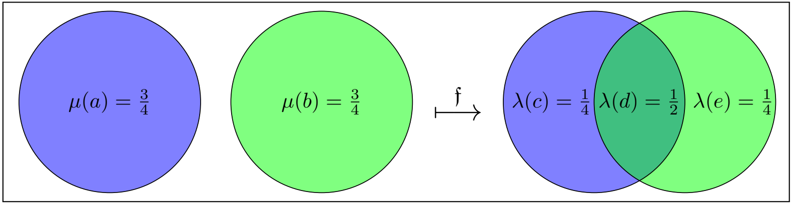

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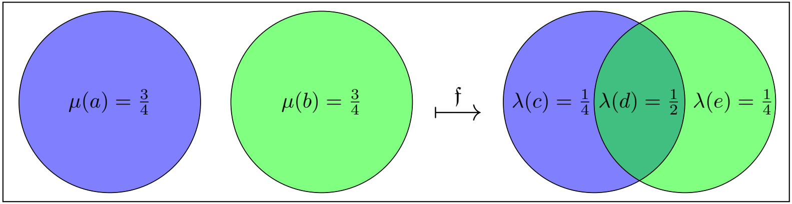

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- If we want it to be additive and maintain these values, we will need a and b to intersect.
- So we arrive at the Boolean algebra $\mathfrak{B}$ of three atoms c, d and e and the measure $\lambda : \mathfrak{B} \to \mathbb{R}$ defined by

$$\lambda(c) = \lambda(e) = \frac{1}{4} \text{ and } \lambda(d) = \frac{1}{2}.$$

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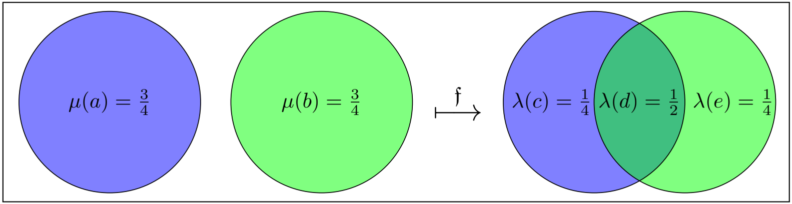

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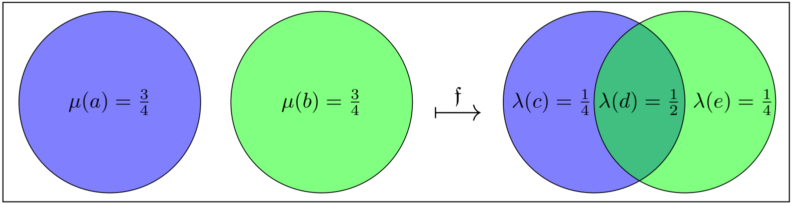

We are in fact solving the following system of linear equations:

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- $\mu(a) = \lambda(c) + \lambda(d)$;
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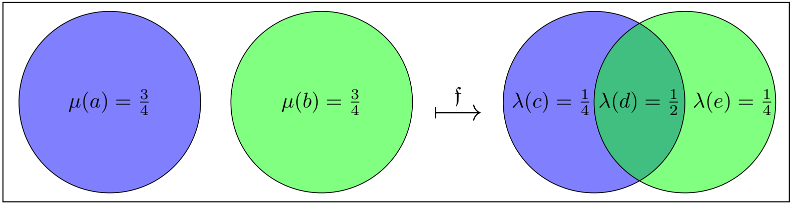

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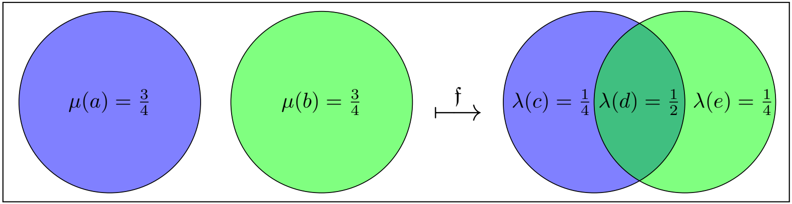

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The final construction is obtained as a direct limit of these finite constructions.

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- Now define $\mathfrak{f}$ by

$$\mathfrak{f}([s]) = \bigcup\{[t] : s \text{ generates } t\}.$$

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- Let $g \in T^{(n)}$ be generated by precisely the members of B.

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Note that as it is defined, (T.1) fails (the image of this map does not generate $\mathfrak{B}$). But just consider the algebra generated by this image.

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Warning! If each $X_i = \{1, 2\}$ then μ is not exhaustive.

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Lemma

If μ is a submeasure and the corresponding signed measure is non-negative, then μ must dominate a non-trivial measure (i.e. it cannot be **pathological***).*

In particular the signed measure corresponding to Talagrand's submeasure is indeed non-negative.

On the other hand, there are very simple submeasures where the corresponding signed measure is unbounded. For example take the submeasure

$$\mu(a) = \begin{cases} 1, & \text{if } a = 1; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

The End