Wallman representations of hyperspaces

Wojciech Stadnicki (University of Wrocław)

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We say $X$ is a $C$-space (or $X$ has property $C$) if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of $X$, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, such that:

- each $\mathcal{V}_i$ is a family of pairwise disjoint open subsets of $X$
- $\mathcal{V}_i \prec \mathcal{U}_i$ ({$\mathcal{V}_i$} refines $\mathcal{U}_i$, i.e. $\forall V \in \mathcal{V}_i \exists U \in \mathcal{U}_i$ $V \subseteq U$)
- $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of $X$
C-spaces

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finite dimension $\Rightarrow$ property $C$ $\Rightarrow$ weakly infinite dimension
Theorem (Levin, Rogers 1999)

If $X$ is a metric continuum of dimension $\geq 2$ then its hyperspace $C(X)$ is not a $C$-space.
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Theorem
Suppose \( X \) is a 1-dimensional hereditarily indecomposable metric continuum. Then either \( \dim C(X) = 2 \) or \( C(X) \) is not a \( C \)-space.

Question
Are above theorems true for non-metric continua?
Answer: Yes.
Reduce the non-metric case to the metric one by applying L"owenheim-Skolem theorem. Then use the already known theorems. This approach was presented by K. P. Hart on the Winter School in 2012.
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Each (distributive and separative) lattice $L$ corresponds to the Wallman space $wL$, which consists of all ultrafilters on $L$. 

For $a \in L$ let \( \hat{a} = \{ u \in wL : a \in u \} \). We define the topology in $wL$ taking the family \( \{ \hat{a} : a \in L \} \) as a base for closed sets.

If $L$ is a countable (normal) lattice then $wL$ is a compact metric space.

Fact

Let $L$ be a sublattice of $2^X$. The function $q : X \to wL$ given by $q(x) = \{ a \in L : x \in a \}$ is a continuous surjection.
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**Fact**

Let $L$ be a sublattice of $2^X$. The function $q : X \to wL$ given by $q(x) = \{ a \in L : x \in a \}$ is a continuous surjection.
Definition

A property \( \mathcal{P} \) is \textit{elementarily reflected} if:
for any compact space \( X \) with the property \( \mathcal{P} \) and for any \( L \prec 2^X \)
its Wallman representation \( wL \) also has \( \mathcal{P} \).
Definition

A property $\mathcal{P}$ is *elementarily reflected* if:
for any compact space $X$ with the property $\mathcal{P}$ and for any $L \prec 2^X$ its Wallman representation $wL$ also has $\mathcal{P}$.

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A property $\mathcal{P}$ is *elementarily reflected by submodels* if:
for any compact space $X$ with the property $\mathcal{P}$ and for any $L \prec 2^X$ of the form $L = 2^X \cap \mathcal{M}$, where $2^X \in \mathcal{M}$ and $\mathcal{M} \prec H(\kappa)$ (for a large enough regular $\kappa$), its Wallman representation $wL$ also has $\mathcal{P}$.
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- Connectedness is elementarily reflected.
- The dimension dim is elementarily reflected (including $\text{dim} = \infty$).
- Hereditary indecomposability is elementarily reflected.
If $\dim X \geq 2$ then $C(X)$ is not a $C$-space:
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Suppose $\dim X \geq 2$. Take countable $\mathcal{M} \prec H(\kappa)$ such that $2^X, 2^{C(X)} \in \mathcal{M}$.

Let $L = 2^X \cap \mathcal{M}$ and $L^* = 2^{C(X)} \cap \mathcal{M}$. Then $wL, wL^*$ are metric continua. Moreover, $\dim wL = \dim X \geq 2$.

By the result of M. Levin and J. T. Rogers, Jr. for metric continua, we obtain $C(wL)$ is not a $C$-space.

**Lemma 1** The space $wL^*$ is homeomorphic to $C(wL)$.

**Property C** is elementarily reflected.

By Lemma 1, $wL^*$ is not a $C$-space. By Lemma (2), neither is $C(X)$.
If $\text{dim}X \geq 2$ then $C(X)$ is not a $C$-space:

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Lemma

1. The space $wL^*$ is homeomorphic to $C(wL)$.
2. Property $C$ is elementarily reflected.
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Lemma

1. The space $wL^*$ is homeomorphic to $C(wL)$.
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By Lemma (1) $wL^*$ is not a $C$-space.
By Lemma (2), neither is $C(X)$. 
Property C is elementarily reflected \((proof)\):

Let \(X\) be a \(C\)-space, 2\(\mathcal{X}\) the lattice of its closed subsets and \(\mathcal{L} \prec 2\mathcal{X}\).

Suppose \(U_1, U_2, \ldots\) is a sequence of finite open covers of \(\mathcal{wL}\), consisting of basic sets (i.e. for all \(U_{ik} \in U_i\) there is \(F_{ik} \in \mathcal{L}\) such that \(U_{ik} = \mathcal{wL} \\setminus \hat{F}_{ik}\)).

Define \(U'_{ik} = X \setminus F_{ik}\) and \(U'_i = \{U'_{i1}, U'_{i2}, \ldots, U'_{ik_i}\}\). Then \(U'_1, U'_2, \ldots\) is a sequence of open covers of \(X\).

Hence, there exists a finite sequence \(V'_1, V'_2, \ldots, V'_n\) of finite families as in the definition of a \(C\)-space.

So we have:

\[
\mathcal{L} = \exists G_{11}, \ldots, G_{1m_1}, G_{21}, \ldots, G_{2m_2}, \ldots, G_{n1}, \ldots, G_{nm_n} \text{ such that:} \]

1. \(\bigwedge_{n_i=1}^{\infty} (\bigwedge_{j \leq j' < m_i} (G_{ij} \cup G_{ij'} = X))\)
2. \(\bigwedge_{n_i=1}^{\infty} (\bigvee_{m_i=1}^{\infty} (\bigwedge_{k_i=1}^{k_i} (G_{ij} \cap F_{ij'} = F_{ij'})))\)
3. \(\bigwedge_{n_i=1}^{\infty} \bigwedge_{m_i=1}^{\infty} G_{ij} = \emptyset\).

By elementarity such sets \(G_{ij}\) exist in \(\mathcal{L}\).

Take \(V_{ij} = \mathcal{wL} \setminus \hat{G}_{ij}\) and \(V_i = \{V_{i1}, V_{i2}, \ldots, V_{im_k}\}\). Then \(V_1, V_2, \ldots, V_n\) are families of pairwise disjoint sets (by (1)), open in \(\mathcal{wL}\). For \(i \leq n\) the family \(V_i\) refines \(U_i\) (by (2)) and \(\bigcup_{n_i=1}^{\infty} V_i\) is a cover of \(\mathcal{wL}\) (by (3)).
Property C is elementarily reflected (proof):

Let $X$ be a $C$-space, $2^X$ the lattice of its closed subsets and $L \prec 2^X$.

Suppose $U_1, U_2, \ldots$ is a sequence of finite open covers of $\omega L$, consisting of basic sets (i.e. for all $U_{ik} \in U_i$ there is $F_{ik} \in L$ such that $U_{ik} = \omega L \setminus \hat{F}_{ik}$). Define $U'_{ik} = X \setminus F_{ik}$ and $U'_i = \{U'_{i1}, U'_{i2}, \ldots, U'_{ik_i}\}$. Then $U'_1, U'_2, \ldots$ is a sequence of open covers of $X$.

Hence, there exists a finite sequence $V'_{11}, V'_{12}, \ldots, V'_{nm}$ of finite families as in the definition of a $C$-space. So we have:

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3. $\bigwedge_{n_i=1}^{\bigwedge_n} \bigwedge_{m_i=1}^{\bigwedge_m} G_{ij} = \emptyset$.

By elementarity such sets $G_{ij}$ exist in $L$. Take $V_{ij} = \omega L \setminus \hat{G}_{ij}$ and $V_i = \{V_{i1}, V_{i2}, \ldots, V_{im_i}\}$. Then $V_1, V_2, \ldots, V_n$ are families of pairwise disjoint sets (by (1)), open in $\omega L$. For $i \leq n$ the family $V_i$ refines $U_i$ (by (2)) and $\bigcup_{n_i=1}^{\bigcup_n} V_i$ is a cover of $\omega L$ (by (3)).
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Define $U_i'_{ik} = X \setminus F_{ik}$ and $U_i' = \{U_i'_{i1}, U_i'_{i2}, \ldots, U_i'_{im_i}\}$. Then $U_i', U_2', \ldots$ is a sequence of open covers of $X$.

Hence, there exists a finite sequence $V_i', V_2', \ldots, V_n'$ of finite families as in the definition of a $C$-space. So we have:

$$2^X = \exists G_{11}, \ldots, G_{m_1}, G_{21}, \ldots, G_{m_2}, \ldots, G_{n1}, \ldots, G_{nm}$$

such that:

1. $\bigwedge_{i=1}^n (\bigwedge_{j=j_i}^{j_i'} (G_{ij} \cup G_{ij}')) = X$
2. $\bigwedge_{i=1}^n (\bigwedge_{m=1}^{m_i} G_{ij} \cap F_{ij}') = F_{ij}'$
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Property $C$ is elementarily reflected *(proof)*:

Let $X$ be a $C$-space, $2^X$ the lattice of its closed subsets and $L \vartriangleleft 2^X$. Suppose $U_1, U_2, \ldots$ is a sequence of finite open covers of $wL$, consisting of basic sets (i.e. for all $U_{ik} \in U_i$ there is $F_{ik} \in L$ such that $U_{ik} = wL \setminus \hat{F}_{ik}$). Define $U'_{ik} = X \setminus F_{ik}$ and $U'_i = \{U'_{i1}, U'_{i2}, \ldots, U'_{ik_i}\}$. Then $U'_1, U'_2, \ldots$ is a sequence of open covers of $X$. 

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Let $X$ be a $C$-space, $2^X$ the lattice of its closed subsets and $L \ll 2^X$. Suppose $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a sequence of finite open covers of $wL$, consisting of basic sets (i.e. for all $U_{ik} \in \mathcal{U}_i$ there is $F_{ik} \in L$ such that $U_{ik} = wL \setminus \hat{F}_{ik}$). Define $U'_{ik} = X \setminus F_{ik}$ and $\mathcal{U}'_i = \{U'_{i1}, U'_{i2}, \ldots, U'_{ik_i}\}$. Then $\mathcal{U}'_1, \mathcal{U}'_2, \ldots$ is a sequence of open covers of $X$. Hence, there exists a finite sequence $\mathcal{V}'_1, \mathcal{V}'_2, \ldots, \mathcal{V}'_n$ of finite families as in the definition of a $C$-space.
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$2^X \models \exists G_{11}, \ldots, G_{1m_1}, G_{21}, \ldots G_{2m_2}, \ldots, G_{n1}, \ldots, G_{nm_n}$ such that:

1. $\bigwedge_{i=1}^{n} \left( \bigwedge_{1 \leq j < j' \leq m_i} \left( G_{ij} \cup G_{ij'} = X \right) \right)$
2. $\bigwedge_{i=1}^{n} \left( \bigwedge_{j=1}^{m_i} \left( \bigvee_{j'=1}^{k_i} \left( G_{ij} \cap F_{ij'} = F_{ij'} \right) \right) \right)$
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The space $wL^*$ is homeomorphic to $C(wL)$ (sketch of proof):

Let $u^* \in wL^*$. Extend it to an ultrafilter $u$ on $2^{C(X)}$. Let $K_u \in C(X)$ be the only point in $\bigcap u$. So $K_u$ is a subcontinuum of $X$. Define $h(u^*) = q[K_u]$, where $q: X \to wL$ is the continuous surjection given by $q(x) = \{ a \in L : x \in a \}$. Then $h$ does not depend on the choice of $K_u$ and it is a homeomorphism.
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Then $h$ does not depend on the choice of $K_u$ and it is a homeomorphism.
Proposition

Being not a $C$-space is elementarily reflected by submodels.

Let $X$ be a non-$C$-space, $M≺H(\kappa)$, such that $2^X ∈ M$ and $L = 2^X \cap M$.

There is a sequence $(U_i)_{i=1}^{∞} ∈ H(\kappa)$ witnessing that $X$ is not a $C$-space.

Hence, $H(\kappa)$ models the following sentence $ϕ$:

There exists a sequence $(F_i)_{i=1}^{∞}$ of finite subsets of $2^X$ such that

$\bigcap F_i = \emptyset$ for each $i$ and for no $m ∈ N$ and $G_1, \ldots, G_m$ finite subsets of $2^X$, the following conditions hold simultaneously:

- for $j ≤ m$ and distinct $G$, $G' \in G_j$ their union $G \cup G' = X$,
- for $j ≤ m$ and $G \in G_j$ there exists $F \in F_j$ such that $F \subseteq G$,

$\bigcap (G_1 \cup \ldots \cup G_m) = \emptyset$.

By elementarity $M| = ϕ$. Therefore such a sequence $(F_i)_{i=1}^{∞}$ exists in $M$.

Each $F_i$ is finite, so $F_i \subseteq L$. Define $U'_i = \{ w ∈ L : F \in F_i \}$.

The sequence $(U'_i)_{i=1}^{∞}$ witnesses that $wL$ is not a $C$-space.
Proposition

Being not a $C$-space is elementarily reflected by submodels.

Let $X$ be a non-$C$-space, $\mathcal{M} \prec H(\kappa)$, such that $2^X \in \mathcal{M}$ and $L = 2^X \cap \mathcal{M}$. There is a sequence $(U_i)_{i=1}^\infty \in H(\kappa)$ witnessing that $X$ is not a $C$-space. Hence, $H(\kappa)$ models the following sentence $\phi$:

There exists a sequence $(F_i)_{i=1}^\infty$ of finite subsets of $2^X$ such that $\bigcap F_i = \emptyset$ for each $i$ and for no $m \in \mathbb{N}$ and $G_1, \ldots, G_m$ finite subsets of $2^X$, the following conditions hold simultaneously:

- For $j \leq m$ and distinct $G, G' \in G_j$ their union $G \cup G' = X$,
- For $j \leq m$ and $G \in G_j$ there exists $F \in F_j$ such that $F \subseteq G$.

$\bigcap (G_1 \cup \ldots \cup G_m) = \emptyset$.

By elementarity $M \models \phi$. Therefore such a sequence $(F_i)_{i=1}^\infty$ exists in $M$. Each $F_i$ is finite, so $F_i \subseteq L$. Define $U'_i = \{ wL \setminus \hat{F} : F \in F_i \}$.

The sequence $(U'_i)_{i=1}^\infty$ witnesses that $wL$ is not a $C$-space.
Proposition

Being not a $C$-space is elementarily reflected by submodels.

Let $X$ be a non-$C$-space, $\mathcal{M} \prec H(\kappa)$, such that $2^X \in \mathcal{M}$ and $L = 2^X \cap \mathcal{M}$. There is a sequence $(U_i)_{i=1}^\infty \in H(\kappa)$ witnessing that $X$ is not a $C$-space. Hence, $H(\kappa)$ models the following sentence $\varphi$: 

\[
\text{There exists a sequence} (F_i)_{i=1}^\infty \text{of finite subsets of } 2^X \text{ such that } \bigcap F_i = \emptyset \text{ for each } i \text{ and for no } m \in \mathbb{N} \text{ and } G_1, \ldots, G_m \text{ finite subsets of } 2^X, \text{ the following conditions hold simultaneously: for } j \leq m \text{ and distinct } G, G' \in G_j \text{ their union } G \cup G' = X, \text{ for } j \leq m \text{ and } G \in G_j \text{ there exists } F \in F_j \text{ such that } F \subseteq G, \text{ } \bigcap (G_1 \cup \ldots \cup G_m) = \emptyset.
\]
Proposition

Being not a $C$-space is elementarily reflected by submodels.

Let $X$ be a non-$C$-space, $\mathcal{M} \prec H(\kappa)$, such that $2^X \in \mathcal{M}$ and $L = 2^X \cap \mathcal{M}$. There is a sequence $(\mathcal{U}_i)_{i=1}^\infty \in H(\kappa)$ witnessing that $X$ is not a $C$-space. Hence, $H(\kappa)$ models the following sentence $\varphi$:

There exists a sequence $(\mathcal{F}_i)_{i=1}^\infty$ of finite subsets of $2^X$ such that $\bigcap \mathcal{F}_i = \emptyset$ for each $i$ and for no $m \in \mathbb{N}$ and $G_1, \ldots, G_m$ finite subsets of $2^X$, the following conditions hold simultaneously:

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Proposition

Being not a $C$-space is elementarily reflected by submodels.

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By elementarity $M \models \varphi$. Therefore such a sequence $(F_i)_{i=1}^\infty$ exists in $M$. Each $F_i$ is finite, so $F_i \subseteq L$. Define $U'_i = \{ wL \setminus \hat{F} : F \in F_i \}$. The sequence $(U'_i)_{i=1}^\infty$ witnesses that $wL$ is not a $C$-space.
Fact

A normal space $X$ is weakly infinite dimensional if and only if it is a 2-$C$-space.

Definition

For $m \geq 2$ we say $X$ is an $m$-$C$-space if for each sequence $U_1, U_2, \ldots$ of open covers of $X$ such that $|U_i| \leq m$, there exists a sequence $V_1, V_2, \ldots$, such that:

- each $V_i$ is a family of pairwise disjoint open subsets of $X$
- $V_i \prec U_i$ ($V_i$ refines $U_i$, i.e. $\forall V \in V_i \exists U \in U_i \ V \subseteq U$)
- $\bigcup_{i=1}^{\infty} V_i$ is a cover of $X$
2-C-spaces ⊇ 3-C-spaces ⊇ \ldots ⊇ n-C-spaces ⊇ \ldots ⊇ C-spaces
2-\(C\)-spaces \(\supseteq\) 3-\(C\)-spaces \(\supseteq\) \ldots \(\supseteq\) \(n\)-\(C\)-spaces \(\supseteq\) \ldots \(\supseteq\) \(C\)-spaces

**Corollary**

- Weak infinite dimension is elementarily reflected.
- Strong infinite dimension is elementarily reflected by submodels.
2-C-spaces $\supseteq$ 3-C-spaces $\supseteq$ ... $\supseteq$ n-C-spaces $\supseteq$ ... $\supseteq$ C-spaces

**Corollary**
- Weak infinite dimension is elementarily reflected.
- Strong infinite dimension is elementarily reflected by submodels.

**Corollary**
If there exist a compact space which is weakly infinite dimensional but fails to be a C-space, then there exists such a space which is metric.