Wallman representations of hyperspaces

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C-spaces

We say X is a C-space (or X has property C) if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, such that:

- each $\mathcal{V}_i$ is a family of pairwise disjoint open subsets of X
- $\mathcal{V}_i \prec \mathcal{U}_i$ ($\mathcal{V}_i$ refines $\mathcal{U}_i$, i.e. $\forall\ V \in \mathcal{V}_i\ \exists U \in \mathcal{U}_i\ V \subseteq U$)
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finite dimension $\Rightarrow$ property C $\Rightarrow$ weakly infinite dimension

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Are above theorems true for non-metric continua?

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Question

Are above theorems true for non-metric continua?
Answer: Yes.

Reduce the non-metric case to the metric one by applying Löwenheim-Skolem teorem. Then use the already known theorems. This approach was presented by K. P. Hart on the Winter School in 2012.

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If L is a countable (normal) lattice then wL is a compact metric space.

Fact

Let L be a sublattice of 2^X. The function $q\colon X \to wL$ given by $q(x) = \{a \in L\colon x \in a\}$ is a continuous surjection.

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for any compact space X with the property $\mathcal{P}$ and for any $L \prec 2^X$ of the form $L = 2^X \cap \mathcal{M}$, where $2^X \in \mathcal{M}$ and $\mathcal{M} \prec H(\kappa)$ (for a large enough regular κ), its Wallman representation wL also has $\mathcal{P}$.

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- Connectedness is elementarily reflected.
- The dimension dim is elementarily reflected (including $\dim = \infty$).
- Hereditary indecomposability is elementarily reflected.

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By the result of M. Levin and J. T. Rogers, Jr. for metric continua, we obtain $C(wL)$ is not a C-space.

Lemma

1. The space wL^* is homeomorphic to $C(wL)$.
2. Property C is elementarily reflected.

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1. The space wL^* is homeomorphic to $C(wL)$.
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By Lemma (1) wL^* is not a C-space.
By Lemma (2), neither is $C(X)$.

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$2^X \models \exists G_{11}, \ldots, G_{1m_1}, G_{21}, \ldots G_{2m_2}, \ldots, G_{n1}, \ldots, G_{nm_n}$ such that:

(1) $\bigwedge_{i=1}^{n} \left(\bigwedge_{1 \leq j < j' \leq m_i} \left(G_{ij} \cup G_{ij'} = X \right) \right)$

(2) $\bigwedge_{i=1}^{n} \left(\bigwedge_{j=1}^{m_i} \left(\bigvee_{j'=1}^{k_i} \left(G_{ij} \cap F_{ij'} = F_{ij'} \right) \right) \right)$

(3) $\bigcap_{i=1}^{n} \bigcap_{j=1}^{m_i} G_{ij} = \emptyset$.

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By elementarity such sets G_{ij} exist in L. Take $V_{ij} = wL \setminus \widehat{G_{ij}}$ and $\mathcal{V}_i = \{V_{i1}, V_{i2}, \ldots, V_{im_k}\}$. Then $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ are families of pairwise disjoint sets (by (1)), open in wL. For $i \le n$ the family $\mathcal{V}_i$ refines $\mathcal{U}_i$ (by (2)) and $\bigcup_{i=1}^n \mathcal{V}_i$ is a cover of wL (by (3)).

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Define $h(u^*) = q[K_u]$, where $q\colon X \to wL$ is the continuous surjection given by $q(x) = \{a \in L\colon x \in a\}$.

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Then h does not depend on the choice of K_u and it is a homeomorphism.

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- *for $j \leq m$ and distinct $G, G' \in \mathcal{G}_j$ their union $G \cup G' = X$,*
- *for $j \leq m$ and $G \in \mathcal{G}_j$ there exists $F \in \mathcal{F}_j$ such that $F \subseteq G$,*
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By elementarity $\mathcal{M} \models \varphi$. Therefore such a sequence $(\mathcal{F}_i)_{i=1}^{\infty}$ exists in $\mathcal{M}$. Each $\mathcal{F}_i$ is finite, so $\mathcal{F}_i \subseteq L$. Define $\mathcal{U}'_i = \{wL \setminus \widehat{F} \colon F \in \mathcal{F}_i\}$. The sequence $(\mathcal{U}'_i)_{i=1}^{\infty}$ witnesses that wL is not a C-space.

Fact

A normal space X is weakly infinite dimensional if and only if it is a 2-C-space.

Definition

For $m \geq 2$ we say X is an m-C-space if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X such that $|\mathcal{U}_i| \leq m$, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, such that:

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Corollary

- Weak infinite dimension is elementarily reflected.
- Strong infinite dimension is elementarily reflected by submodels.

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If there exist a compact space which is weakly infinite dimensional but fails to be a C-space, then there exists such a space which is metric.