Universal structures on the Urysohn universal space

Michal Doucha

Charles University in Prague, Institute of Mathematics, AS

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- Any finite partial isometry $\phi : \{x_1, \ldots, x_n\} \subseteq \mathbb{U} \rightarrow \{y_1, \ldots, y_n\} \subseteq \mathbb{U}$ can be extended to an isometry $\bar{\phi} \supseteq \phi : \mathbb{U} \rightarrow \mathbb{U}$ on the whole space.

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Fact

There is only one such a space up to isometry and it contains an isometric copy of every separable metric space.

General goal

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Examples of Polish structures:

- Polish metric spaces equipped with finitely or countably many closed relations (i.e. closed subsets of the space or its products)

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- Polish metric spaces equipped with a continuous function to some fixed space

Coding of Polish spaces and Polish structures

In order to use the methods of descriptive set theory in investigating/classifying some class of mathematical structures one needs to find a way how to code this class as a standard Borel space. The Effros-Borel space (of some Polish space) can sometimes serve in this direction.

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In order to use the methods of descriptive set theory in investigating/classifying some class of mathematical structures one needs to find a way how to code this class as a standard Borel space. The Effros-Borel space (of some Polish space) can sometimes serve in this direction.

Let X be a Polish space and $F(X)$ the set of all closed subsets of X. Let $\mathcal{B}$ be a σ-algebra on $F(X)$ generated by the sets $\{F \in F(X) : F \cap U \neq \emptyset \wedge U$ is a basic open set of $X\}$. $(F(X), \mathcal{B})$ is then a standard Borel space called the Effros-Borel space of $F(X)$.

Coding of Polish spaces and Polish structures-examples

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- $F(\mathbb{U})$ - coding of all Polish metric spaces

Main results

Theorem

Let $n_1 \leq \ldots \leq n_m$ be a finite non-decreasing sequence of natural numbers. For every $1 \leq i \leq m$ there is a closed set $F_{n_i} \subseteq \mathbb{U}^{n_i}$ such that for any Polish metric space (X, d) equiped with closed sets $G_{n_i} \subseteq X^{n_i}$ there is an isometry $\psi : X \hookrightarrow \mathbb{U}$ such that for all $i \leq m$ $\psi^{n_i}(X^{n_i}) \cap F_{n_i} = \psi^{n_i}(G_{n_i})$.

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There is also a version with infinitely many closed relations that is slightly weaker though.

Main results

Theorem

Let $\mathbb{U}$ be the universal Urysohn space. For every $n, m \in \mathbb{N}$ there exist closed sets F_m^n such that $F_m^n \subseteq \mathbb{U}^n$ which are universal in the following sense. Let (X, d) be a Polish metric space equipped with closed sets G_m^n, for all $m, n \in \mathbb{N}$, where $G_m^n \subseteq X^n$. Then there exist an isometric embedding $\psi : X \hookrightarrow \mathbb{U}$ and injections $\pi_n : \mathbb{N} \to \mathbb{N}$ for all $n \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}(\psi^n(X^n) \cap F^n_{\pi_n(m)} = \psi^n(G^n_m))$.

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There exist a closed set $C \subseteq \mathbb{U} \times [0,1]$ *such that for any Polish metric space* (X, d) *and closed set* $B \subseteq X \times [0,1]$ *there exists an isometric emebedding* $\psi : X \to \mathbb{U}$ *that moreover respects* B*; i.e.* $\psi(X) \times [0,1] \cap C = \tilde{\psi}(B)$*, where* $\tilde{\psi}(x, r) = (\psi(x), r)$ *for* $(x, r) \in X \times [0,1]$.

Sketch of a simple case

Let us consider the following simple version of the main theorem.

Theorem

There is a closed set $F_{\mathbb{U}} \subseteq \mathbb{U}^2$ *such that for any Polish metric space* (X, d) *equipped with a closed set* $F_X \subseteq X^2$ *there is an isometry* $\psi : X \hookrightarrow \mathbb{U}$ *such that* $\psi^2(X) \cap F_{\mathbb{U}} = \psi^2(F_X)$.

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We use Fraïssé theory to find such a set.
We describe a class of finite structures, prove that it is a Fraïssé class and its Fraïssé limit is the Urysohn universal space along with the universal closed set in the square.

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- There is a rational function $p : A^2 \to \mathbb{Q}_0^+$ such that $\forall (a_1, a_2), (b_1, b_2) \in A^2 (p(a_1, a_2) \leq p(b_1, b_2) + d(a_1, b_1) + d(a_2, b_2))$.

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The interpretation of the function p is that it gives to a pair of points a distance (in the sum metric) from the desired closed set F.

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4. It satisfies the amalgamation property; i.e. for any structures $A, B, C \in \mathcal{K}$ such that A is embedded into B via ϕ_A and into C via ϕ_C there is some D such that both B and C embedded into D via ψ_B, resp. ψ_C so that $\psi_B \circ \phi_B = \psi_C \circ \phi_C$.

Sketch of a simple case

Proof of 4.

Suppose A is a substructure of both B and C. We set the underlying set for D to be $A \coprod (B \setminus A) \coprod (C \setminus A)$. We extend the metric as usual: for $b \in B$ and $c \in C$ we set $d(b, c) = \min\{d(b, a) + d(a, c) : a \in A\}$. And for $b \in B$ and $c \in C$ we set $p(b, c) = \max\{|p(x, y) - (d(x, b) + d(y, c))| : (x, y) \in B^2 \cup C^2\}$.

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Let $\vec{a}, \vec{b} \in D^2$ be given.
Suppose that $p(\vec{a}) = p(\vec{x}) - d(\vec{a}, \vec{x})$ for some $\vec{x} \in B^2 \cup C^2$. Then $p(\vec{a}) = p(\vec{x}) - d(\vec{a}, \vec{x}) \leq p(\vec{x}) - d(\vec{b}, \vec{x}) + d(\vec{a}, \vec{b}) \leq p(\vec{b}) + d(\vec{a}, \vec{b})$.

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Suppose that $p(\vec{a}) = d(\vec{a},\vec{x}) - p(\vec{x})$ for some $\vec{x} \in B^2 \cup C^2$. Then $p(\vec{a}) = d(\vec{a},\vec{x}) - p(\vec{x}) \leq d(\vec{a},\vec{b}) + d(\vec{b},\vec{x}) - p(\vec{x}) \leq d(\vec{a},\vec{b}) + p(\vec{b})$. □

Sketch of a simple case

So $\mathcal{K}$ has a Fraïssé limit which we will denote U (we do not notationally distinguish between the structure and its underlying set) and it is the rational Urysohn metric space along with a closed set $F' \subseteq U^2$ defined as follows: for $(u_1, u_2) \in U^2$ we have $(u_1, u_2) \in F'$ iff $p(u_1, u_2) = 0$. Let $\mathbb{U}$ be the completion of U and $F_{\mathbb{U}} \subseteq \mathbb{U}$ the closure of F' in $\mathbb{U}$.

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Let (X, d) be a Polish metric space equipped with a closed set $F_X \in X^2$. Let $\{d_i : i \in \mathbb{N}\} \subseteq X$ be a countable dense subset. There exists an isometry $\tilde{\phi} : \{d_i : i \in \mathbb{N}\} \to \{u_i : i \in \mathbb{N}\}$ sending d_i to u_i, for $i \in \mathbb{N}$, such that for all $i, j \in \mathbb{N}$
$d_{\mathbb{U}}((u_i, u_j), F_{\mathbb{U}}) \approx d((d_i, d_j), F_X)/2$.
We can then extend the isometry $\tilde{\phi}$ to $\tilde{\phi} \subseteq \phi : X \hookrightarrow \mathbb{U}$ and that is it.

Sketch of a simple case

Let $(x_1, x_2) \in X^2$ be arbitrary.

- Suppose that $(x_1, x_2) \notin F_X$ and let $\varepsilon = d((x_1, x_2), F_X)$. There exist $(d_i, d_j) \in D^2$ such that $d((d_1, d_2), (x_1, x_2)) < \varepsilon/3$. It follows that $d((d_i, d_j), F_X) > 2\varepsilon/3$, thus $d_{\mathbb{U}}((u_i, u_j), F_{\mathbb{U}}) > \varepsilon/3$, thus $d_{\mathbb{U}}((\phi(x_1), \phi(x_2)), F_{\mathbb{U}}) > 0$, i.e. $(\phi(x_1), \phi(x_2)) \notin F_{\mathbb{U}}$.

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- Suppose that $(x_1, x_2) \in F_X$ but $d_{\mathbb{U}}((\phi(x_1), \phi(x_2)), F_{\mathbb{U}}) > 0$. Then we would again find $(u_i, u_j) \in U^2$ such that $d_{\mathbb{U}}((u_i, u_j), F_{\mathbb{U}}) = \varepsilon > 0$ and $d_{\mathbb{U}}((u_i, u_j), (\phi(x_1), \phi(x_2))) < \varepsilon$. But then $d((d_i, d_j), F_X) \geq \varepsilon$ and $d((d_i, d_j), (x_1, x_2)) < \varepsilon$, so $(x_1, x_2) \notin F_X$, a contradiction.

Classification of Polish metric spaces

Let us consider the class of Polish metric spaces (coded by $F(\mathbb{U})$) with the relation of isometry.
We cannot in general extend an isometry $\phi : X \subseteq \mathbb{U} \to Y \subseteq \mathbb{U}$ to an isometry $\phi \subseteq \bar{\phi} : \mathbb{U} \to \mathbb{U}$. However, there is the following theorem.

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Theorem (Gao-Kechris; 2003)

Let E_I be an equivalence relation on $F(\mathbb{U})$ such that for $X, Y \in F(\mathbb{U})$ XE_iY iff X and Y are isometric, and let F_I be an equivalence relation on $F(\mathbb{U})$ induced by a group action of $\mathrm{Iso}(\mathbb{U})$, i.e. for $X, Y \in F(\mathbb{U})$ XF_IY iff there exists an isometry $\phi : \mathbb{U} \to \mathbb{U}$ such that $\phi[X] = Y$.
Then $E_I \leq_B F_I$.