Universal structures on the Urysohn universal space

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The Urysohn universal metric space $\mathbb{U}$ is a Polish metric space which is both universal and homogeneous for the class of all finite metric spaces; i.e. It contains an isometric copy of any finite metric space. Any finite partial isometry $\phi : \{x_1, \ldots, x_n\} \subseteq \mathbb{U} \rightarrow \{y_1, \ldots, y_n\} \subseteq \mathbb{U}$ can be extended to an isometry $\bar{\phi} \supseteq \phi : \mathbb{U} \rightarrow \mathbb{U}$ on the whole space.

Fact: There is only one such a space up to isometry and it contains an isometric copy of every separable metric space.
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**Fact**

*There is only one such a space up to isometry and it contains an isometric copy of every separable metric space.*
General goal

The goal is to enrich the Urysohn universal space with some additional structure so that this enriched Urysohn space is still universal and homogeneous for that particular kind of Polish structure.
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Examples of Polish structures:

- Polish metric spaces equipped with finitely or countably many closed relations (i.e. closed subsets of the space or its products)
General goal

- Polish metric spaces equipped with closed subsets of the product of the space and some other fixed Polish space ([0, 1]).
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- Polish metric spaces equipped with closed subsets of the product of the space and some other fixed Polish space ([0, 1]).
- Polish metric spaces equipped with a continuous function to some fixed space
In order to use the methods of descriptive set theory in investigating/classifying some class of mathematical structures one needs to find a way how to code this class as a standard Borel space. The Effros-Borel space (of some Polish space) can sometimes serve in this direction.
Coding of Polish spaces and Polish structures

In order to use the methods of descriptive set theory in investigating/classifying some class of mathematical structures one needs to find a way how to code this class as a standard Borel space. The Effros-Borel space (of some Polish space) can sometimes serve in this direction.

Let $X$ be a Polish space and $F(X)$ the set of all closed subsets of $X$. Let $\mathcal{B}$ be a $\sigma$-algebra on $F(X)$ generated by the sets 
$$\{ F \in F(X) : F \cap U \neq \emptyset \land U \text{ is a basic open set of } X \}.$$ 
$(F(X), \mathcal{B})$ is then a standard Borel space called the Effros-Borel space of $F(X)$. 

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Examples of Effros-Borel spaces:

- $F(\mathbb{R}^N)$ - coding of all Polish spaces

- Consider $F(C([0,1]))$ and its Borel subset $\text{Subs}(C([0,1])) = \{ X \in F(C([0,1])) : X \text{ is a closed linear space of } C([0,1]) \}$ - coding of all separable Banach spaces

- $F(U)$ - coding of all Polish metric spaces
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Examples of Effros-Borel spaces:

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- $F(\mathbb{U})$ - coding of all Polish metric spaces
Main results

Theorem

Let \( n_1 \leq \ldots \leq n_m \) be a finite non-decreasing sequence of natural numbers. For every \( 1 \leq i \leq m \) there is a closed set \( F_{n_i} \subseteq \mathbb{U}^{n_i} \) such that for any Polish metric space \((X, d)\) equipped with closed sets \( G_{n_i} \subseteq X^{n_i} \) there is an isometry \( \psi : X \hookrightarrow \mathbb{U} \) such that for all \( i \leq m \)
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\psi_{n_i}(X^{n_i}) \cap F_{n_i} = \psi_{n_i}(G_{n_i}).
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Let $n_1 \leq \ldots \leq n_m$ be a finite non-decreasing sequence of natural numbers. For every $1 \leq i \leq m$ there is a closed set $F_{n_i} \subseteq \mathbb{U}^{n_i}$ such that for any Polish metric space $(X, d)$ equipped with closed sets $G_{n_i} \subseteq X^{n_i}$ there is an isometry $\psi : X \rightarrow \mathbb{U}$ such that for all $i \leq m$

$$\psi^{n_i}(X^{n_i}) \cap F_{n_i} = \psi^{n_i}(G_{n_i}).$$

There is also a version with infinitely many closed relations that is slightly weaker though.
Main results

Theorem

Let $\mathbb{U}$ be the universal Urysohn space. For every $n, m \in \mathbb{N}$ there exist closed sets $F^n_m$ such that $F^n_m \subseteq \mathbb{U}^n$ which are universal in the following sense. Let $(X, d)$ be a Polish metric space equipped with closed sets $G^n_m$, for all $m, n \in \mathbb{N}$, where $G^n_m \subseteq X^n$. Then there exist an isometric embedding $\psi : X \hookrightarrow \mathbb{U}$ and injections $\pi_n : \mathbb{N} \rightarrow \mathbb{N}$ for all $n \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}(\psi^n(X^n) \cap F^n_{\pi_n(m)} = \psi^n(G^n_m))$. 
Main results

Theorem
There exist a closed set $C \subseteq \mathbb{U} \times [0, 1]$ such that for any Polish metric space $(X, d)$ and closed set $B \subseteq X \times [0, 1]$ there exists an isometric emebedding $\psi : X \to \mathbb{U}$ that moreover respects $B$; i.e. $\psi(X) \times [0, 1] \cap C = \tilde{\psi}(B)$, where $\tilde{\psi}(x, r) = (\psi(x), r)$ for $(x, r) \in X \times [0, 1]$. 

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Sketch of a simple case

Let us consider the following simple version of the main theorem.

**Theorem**

There is a closed set $F_U \subseteq U^2$ such that for any Polish metric space $(X, d)$ equipped with a closed set $F_X \subseteq X^2$ there is an isometry $\psi : X \leftrightarrow U$ such that $\psi^2(X) \cap F_U = \psi^2(F_X)$.

We use Fra"ıssé theory to find such a set.

We describe a class of finite structures, prove that it is a Fra"ıssé class and its Fra"ıssé limit is the Urysohn universal space along with the universal closed set in the square.
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We use Fraïssé theory to find such a set.
We describe a class of finite structures, prove that it is a Fraïssé class and its Fraïssé limit is the Urysohn universal space along with the universal closed set in the square.
Definition (The Fraïssé class $\mathcal{K}$)

A finite structure $A$ belongs to $\mathcal{K}$ if

- $A$ is a rational metric space (the metric is denoted $d$).
- There is a rational function $p : A^2 \to \mathbb{Q}^+$ such that $\forall (a_1, a_2), (b_1, b_2) \in A^2 (p(a_1, a_2) \leq p(b_1, b_2) + d(a_1, b_1) + d(a_2, b_2))$.

The interpretation of the function $p$ is that it gives to a pair of points a distance (in the sum metric) from the desired closed set $F$. 

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Sketch of a simple case-definition of the Fraïssé class

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The interpretation of the function $p$ is that it gives to a pair of points a distance (in the sum metric) from the desired closed set $F$. 
Sketch of a simple case

To prove that $\mathcal{K}$ is really a Fraïssé class we need to check that:

1. $\mathcal{K}$ contains only countably many structures (up to isomorphism)
2. It is hereditary, i.e. for each $B \in \mathcal{K}$ and $A$ a substructure of $B$ also $A \in \mathcal{K}$
3. It satisfies the joint embedding property, i.e. for any $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ containing $A$ and $B$ as substructures.
4. It satisfies the amalgamation property; i.e. for any structures $A, B, C \in \mathcal{K}$ such that $A$ is embedded into $B$ via $\phi_A$ and into $C$ via $\phi_C$ there is some $D$ such that both $B$ and $C$ embedded into $D$ via $\psi_B$, resp. $\psi_C$ so that $\psi_B \circ \phi_B = \psi_C \circ \phi_C$. 
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Proof of 4.
Suppose $A$ is a substructure of both $B$ and $C$. We set the underlying set for $D$ to be $A \bigsqcup (B \setminus A) \bigsqcup (C \setminus A)$. We extend the metric as usual: for $b \in B$ and $c \in C$ we set
\[d(b, c) = \min \{d(b, a) + d(a, c) : a \in A\}.\] And for $b \in B$ and $c \in C$ we set
\[p(b, c) = \max \{|p(x, y) - (d(x, b) + d(y, c))| : (x, y) \in B^2 \cup C^2\}.\]
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Let $\vec{a}, \vec{b} \in D^2$ be given.
Suppose that $p(\vec{a}) = p(\vec{x}) - d(\vec{a}, \vec{x})$ for some $\vec{x} \in B^2 \cup C^2$. Then
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p(\vec{a}) = p(\vec{x}) - d(\vec{a}, \vec{x}) \leq p(\vec{x}) - d(\vec{b}, \vec{x}) + d(\vec{a}, \vec{b}) \leq p(\vec{b}) + d(\vec{a}, \vec{b}).\]
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$$p(b, c) = \max\{|p(x, y) - (d(x, b) + d(y, c))| : (x, y) \in B^2 \cup C^2\}.$$ Let $\vec{a}, \vec{b} \in D^2$ be given.
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Suppose that $p(\vec{a}) = d(\vec{a}, \vec{x}) - p(\vec{x})$ for some $\vec{x} \in B^2 \cup C^2$. Then
$$p(\vec{a}) = d(\vec{a}, \vec{x}) - p(\vec{x}) \leq d(\vec{a}, \vec{b}) + d(\vec{b}, \vec{x}) - p(\vec{x}) \leq d(\vec{a}, \vec{b}) + p(\vec{b}).$$
Sketch of a simple case

So $\mathcal{K}$ has a Fraïssé limit which we will denote $U$ (we do not notationally distinguish between the structure and its underlying set) and it is the rational Urysohn metric space along with a closed set $F' \subseteq U^2$ defined as follows: for $(u_1, u_2) \in U^2$ we have $(u_1, u_2) \in F'$ iff $p(u_1, u_2) = 0$. Let $\mathbb{U}$ be the completion of $U$ and $F_U \subseteq \mathbb{U}$ the closure of $F'$ in $\mathbb{U}$. 

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So $\mathcal{K}$ has a Fraïssé limit which we will denote $U$ (we do not notationally distinguish between the structure and its underlying set) and it is the rational Urysohn metric space along with a closed set $F' \subseteq U^2$ defined as follows: for $(u_1, u_2) \in U^2$ we have $(u_1, u_2) \in F'$ iff $p(u_1, u_2) = 0$. Let $\overline{U}$ be the completion of $U$ and $F_U \subseteq \overline{U}$ the closure of $F'$ in $\overline{U}$.

Let $(X, d)$ be a Polish metric space equipped with a closed set $F_X \in X^2$. Let $\{d_i : i \in \mathbb{N}\} \subseteq X$ be a countable dense subset. There exists an isometry $\tilde{\phi} : \{d_i : i \in \mathbb{N}\} \rightarrow \{u_i : i \in \mathbb{N}\}$ sending $d_i$ to $u_i$, for $i \in \mathbb{N}$, such that for all $i, j \in \mathbb{N}$

$$d_U((u_i, u_j), F_U) \approx d((d_i, d_j), F_X)/2.$$ 

We can then extend the isometry $\tilde{\phi}$ to $\tilde{\phi} \subseteq \phi : X \hookrightarrow \overline{U}$ and that is it.
Sketch of a simple case

Let \((x_1, x_2) \in X^2\) be arbitrary.

- Suppose that \((x_1, x_2) \notin F_X\) and let \(\varepsilon = d((x_1, x_2), F_X)\). There exist \((d_i, d_j) \in D^2\) such that \(d((d_1, d_2), (x_1, x_2)) < \varepsilon/3\). It follows that \(d((d_i, d_j), F_X) > 2\varepsilon/3\), thus \(d_U((u_i, u_j), F_U) > \varepsilon/3\), thus \(d_U((\phi(x_1), \phi(x_2)), F_U) > 0\), i.e. \((\phi(x_1), \phi(x_2)) \notin F_U\).
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- Suppose that \((x_1, x_2) \in F_X\) but \(d_U((\phi(x_1), \phi(x_2)), F_U) > 0\). Then we would again find \((u_i, u_j) \in U^2\) such that \(d_U((u_i, u_j), F_U) = \varepsilon > 0\) and \(d_U((u_i, u_j), (\phi(x_1), \phi(x_2))) < \varepsilon\). But then \(d((d_i, d_j), F_X) \geq \varepsilon\) and \(d((d_i, d_j), (x_1, x_2)) < \varepsilon\), so \((x_1, x_2) \notin F_X\), a contradiction.
Classification of Polish metric spaces

Let us consider the class of Polish metric spaces (coded by $F(\mathbb{U})$) with the relation of isometry. We cannot in general extend an isometry $\phi : X \subseteq \mathbb{U} \rightarrow Y \subseteq \mathbb{U}$ to an isometry $\phi \subseteq \overline{\phi} : \mathbb{U} \rightarrow \mathbb{U}$. However, there is the following theorem.

Theorem (Gao-Kechris; 2003)
Let $E_I$ be an equivalence relation on $F(\mathbb{U})$ such that for $X, Y \in F(\mathbb{U})$ $XE_IY$ iff $X$ and $Y$ are isometric, and let $F_I$ be an equivalence relation on $F(\mathbb{U})$ induced by a group action of $\text{Iso}(\mathbb{U})$, i.e. for $X, Y \in F(\mathbb{U})$ $XF_IY$ iff there exists an isometry $\phi : \mathbb{U} \rightarrow \mathbb{U}$ such that $\phi[X] = Y$. Then $E_I \leq B_{F_I}$.
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Then $E_I \leq_B F_I$. 

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