Regularity properties on the real line

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4. februar 2010
Hejnice
Some weak forms of the Axiom of Choice:

- The **Weak Axiom of Choice (wAC)** says that for any countable family of non-empty subsets of a given set of power $2^\aleph_0$ there exists a choice function.

- The **Axiom of Dependent Choice (DC)** says that for any binary relation $R$ on a non-empty set $A$ such that for every $a \in A$ there exists a $b \in A$ such that $aRb$, for every $a \in A$ there exists a function $f : \omega \rightarrow A$ satisfying $f(n)Rf(n + 1)$ for any $n \in \omega$ and $f(0) = a$.

Then

$$AC \rightarrow DC, \quad DC \rightarrow wAC$$

and the implications cannot be reversed.
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and the implications cannot be reversed.
A subset $B \subseteq X$ is called a **Bernstein set** if $|B| = |X \setminus B| = c$ and neither $B$ nor $X \setminus B$ contains a perfect subset.

**Theorem 1 (F. Bernstein [1])**

If an uncountable Polish space $X$ can be well-ordered, then there exists a Bernstein set $B \subseteq X$, i.e. $WR \rightarrow BS$.

- a Bernstein set is a classical example of a non-measurable set

**Theorem 2 (F. Bernstein [1])**

A Bernstein set does not possess the Baire Property and is not Lebesgue measurable, i.e. $BP \rightarrow \neg BS$ and $LM \rightarrow \neg BS$. 
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BS: there exists a Bernstein set

WR: the set of R can be well-ordered

LM: every set of R is Lebesgue measurable

BP: every set of R possesses the Baire property
Let $\langle X, +, 0 \rangle$ be additive group. A set $V \subseteq X$ is called a Vitali set if there exists a countable dense subset $D$ such that

1. $(\forall x, y) ((x, y \in V \land x \neq y) \rightarrow x - y \notin D)$,
2. $(\forall x \in X)(\exists y \in V) x - y \in D$.

Note that, for every $x \in X$ there exists exactly one real $y \in V$ such that $x - y \in D$.

- the family $\{\{y \in X : x - y \in D\} : x \in X\}$ is a decomposition of the set $X$ and we call it the Vitali decomposition
- if there exists a selector for the Vitali decomposition then the selector is a Vitali set
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Theorem 3 (G. Vitali [4])
If the real line can be well-ordered, then there exists a Vitali set, i.e. $\text{WR} \rightarrow \text{VS}$.

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Theorem 4 (G. Vitali [4])
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**LM**: every set of $\mathbb{R}$ is Lebesgue measurable

**BP**: every set of $\mathbb{R}$ possesses the Baire property

**BS**: there exists a Bernstein set

**WR**: the set of $\mathbb{R}$ can be well-ordered

**VS**: there exists a selector for a Vitali set

**Note**: The diagram shows the relationships between these properties, with arrows indicating implications or equivalences between them.
Let us consider the family $\mathcal{P}(\omega)$ of all subsets of $\omega$. $\mathcal{P}(\omega)$ is a Boolean algebra and the set

$$\text{Fin} = \{ A \subseteq \omega : |A| < \aleph_0 \}$$

of all finite subsets of $\omega$ is an ideal of algebra $\mathcal{P}(\omega)$.

- we can consider the quotient algebra $\mathcal{P}(\omega)/\text{Fin}$ and we denote by $t$ its cardinality

$$t = |\mathcal{P}(\omega)/\text{Fin}|$$

- we define relation $\ll$ between cardinalities of sets as

$$|A| \ll |B| \equiv (\exists f) (f: B \overset{\text{onto}}{\rightarrow} A)$$
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$$\mathfrak{t} = \lvert \mathcal{P}(\omega)/\text{Fin} \rvert$$

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$$\lvert A \rvert \ll \lvert B \rvert \equiv (\exists f)(f : B \twoheadrightarrow A)$$
Theorem 5

The inequalities $2^{\aleph_0} \leq \kappa$ and $\kappa \ll 2^{\aleph_0}$ hold true. Moreover, if the set $P(\omega)$ can be well-ordered, then $\kappa = 2^{\aleph_0}$, i.e. $\text{In1} \rightarrow \neg \text{WR}$.

Note the following: if $A, B$ are sets such that $|A| \leq |B|, |B| \ll |A|$ then $A$ can be well-ordered if and only if $B$ can be well-ordered.

Corollary 6

A set of cardinality $\kappa$ can be well-ordered if and only if the set of reals $\mathbb{R}$ can be well-ordered.

Corollary 7

If a set of cardinality $\kappa$ cannot be linearly ordered, then $\aleph_1 < \aleph_1 + c < \aleph_1 + \kappa$, i.e. $\neg \text{Lk} \rightarrow \text{In2}$.
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The inequalities $2^{\aleph_0} \leq \tau$ and $\tau \ll 2^{\aleph_0}$ hold true. Moreover, if the set $\mathcal{P}(\omega)$ can be well-ordered, then $\tau = 2^{\aleph_0}$, i.e. $\ln 1 \rightarrow \neg \text{WR}$.

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If a set of cardinality $\tau$ cannot be linearly ordered, then $\aleph_1 < \aleph_1 + \tau < \aleph_1 + \tau$, i.e. $\neg \text{Lk} \rightarrow \ln 2$. 


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The inequalities $2^{\aleph_0} \leq \mathfrak{c}$ and $\mathfrak{c} \ll 2^{\aleph_0}$ hold true. Moreover, if the set $\mathcal{P}(\omega)$ can be well-ordered, then $\mathfrak{c} = 2^{\aleph_0}$, i.e. $\text{In}1 \rightarrow \neg \text{WR}$.

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The inequalities $2^\aleph_0 \leq \mathfrak{c}$ and $\mathfrak{c} \ll 2^\aleph_0$ hold true. Moreover, if the set $\mathcal{P}(\omega)$ can be well-ordered, then $\mathfrak{c} = 2^\aleph_0$, i.e. $\text{In1} \rightarrow \neg \text{WR}$.

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If a set of cardinality $\mathfrak{c}$ cannot be linearly ordered, then $\aleph_1 < \aleph_1 + c < \aleph_1 + \mathfrak{c}$, i.e. $\neg \text{Lk} \rightarrow \text{In2}$. 
Regularity properties on the real line

- **LM**: every set of $\mathbb{R}$ is Lebesgue measurable
- **BP**: every set of $\mathbb{R}$ possesses the Baire property
- **¬BS**: there exists a Bernstein set
- **WR**: the set of $\mathbb{R}$ can be well-ordered
- **¬Lk**: a set of cardinality $k$ can be linearly ordered
- **¬VS**: there exists a selector for a Vitali set
- **¬Wk**: a set of cardinality $k$ can be well-ordered
- **In1**: $c < k \ll c$
- **In2**: $\aleph_1 < \aleph_1 + c < \aleph_1 + k$

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**VS**: there exists a selector for a Vitali set

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Vitali set $V$ on the Cantor space $\omega^2$

- $(\forall x, y) ((x, y \in V \land x \neq y) \rightarrow \{n : x(n) \neq y(n)\} \in [\omega]^\omega,$
- $(\forall x \in \omega^2) (\exists y \in V) \{n : x(n) \neq y(n)\} \in [\omega]^{<\omega}.$

- The family

$$\{\{y \in \omega^2 : \{n : x(n) \neq y(n)\} \in [\omega]^{<\omega}\} : x \in \omega^2\}$$

is a Vitali decomposition of the Cantor space $\omega^2$

- If $f : \mathcal{P}(\omega) \rightarrow \omega^2$ is a function such that $f(A) = \chi(A)$ for any $A \subseteq \omega$, then

$$\overline{f} : \mathcal{P}(\omega)/\text{Fin} \xrightarrow{1-1 \text{ onto}} \omega^2/\text{Fin}$$
Vitali set \( V \) on the Cantor space \( \omega^2 \)

- \((\forall x, y)((x, y \in V \land x \neq y) \rightarrow \{n : x(n) \neq y(n)\} \in [\omega]^\omega),\)

- \((\forall x \in \omega^2)(\exists y \in V)\{n : x(n) \neq y(n)\} \in [\omega]<\omega\).

- the family

\[
\{\{y \in \omega^2 : \{n : x(n) \neq y(n)\} \in [\omega]<\omega\} : x \in \omega^2\}
\]

is a Vitali decomposition of the Cantor space \( \omega^2 \)

- if \( f : P(\omega) \rightarrow \omega^2 \) is a function such that \( f(A) = \chi(A) \) for any \( A \subseteq \omega \), then

\[
\overline{f} : P(\omega)/\text{Fin} \overset{1-1}{\longrightarrow} \omega^2/\text{Fin} \overset{\text{onto}}{\rightarrow}
\]
Vitali set $V$ on the Cantor space $\omega^2$

- $\forall x, y \in V \land x \neq y \rightarrow \{n : x(n) \neq y(n)\} \in [\omega]^\omega$, 
- $\forall x \in \omega^2 \exists y \in V \{n : x(n) \neq y(n)\} \in [\omega]<^\omega$.

- the family 

$$\{\{y \in \omega^2 : \{n : x(n) \neq y(n)\} \in [\omega]<^\omega\} : x \in \omega^2\}$$

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$$\overline{f} : \mathcal{P}(\omega)/\text{Fin} \overset{1-1}{\rightarrow} \omega^2/\text{Fin}$$
Vitali set $V$ on the Cantor space $\omega^2$

- $(\forall x, y)((x, y \in V \land x \neq y) \rightarrow \{n : x(n) \neq y(n)\} \in [\omega]^{\omega})$,
- $(\forall x \in \omega^2)(\exists y \in V)\{n : x(n) \neq y(n)\} \in [\omega]^{<\omega}$.

- the family

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A Vitali set $V$ on Cantor space $\omega^2$ is a set of cardinality $\mathfrak{c}.$
Vitali set $V$ on the Cantor space $\omega^2$

- $(\forall x, y) ((x, y \in V \land x \neq y) \rightarrow \{n : x(n) \neq y(n)\} \in [\omega]^\omega$,

- $(\forall x \in \omega^2)(\exists y \in V) \{n : x(n) \neq y(n)\} \in [\omega]^{<\omega}$.

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Fact

A Vitali set $V$ on Cantor space $\omega^2$ is a set of cardinality $\mathfrak{t}$. 
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Regularity properties on the real line

Vitali set $V$ on the Cantor space $\omega^2$

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Fact

A Vitali set $V$ on Cantor space $\omega^2$ is a set of cardinality $\mathfrak{c}$. 
Vitali set on the circle $\mathbb{T}$ for the set of all dyadic numbers $\mathbb{D}$

- Vitali decomposition: $\mathbb{T}/\mathbb{D} = \{\{y \in \mathbb{T} : x - y \in \mathbb{D}\} : x \in \mathbb{T}\}$

- $f : \omega^2/\text{Fin} \xrightarrow{1-1} \mathbb{T}/\mathbb{D}$

if there exists a selector for the Vitali decomposition, then a Vitali set is the set of cardinality $\mathfrak{c}$

Vitali set on the circle $\mathbb{T}$ for the set of all rational numbers $\mathbb{Q}$

$\mathbb{T}/\mathbb{Q} \cong (\mathbb{T}/\mathbb{D})/(\mathbb{Q}/\mathbb{D})$

Thus,

$\mathfrak{c} = \aleph_0 \cdot |\mathbb{T}/\mathbb{Q}|$
Vitali set on the circle $\mathbb{T}$ for the set of all dyadic numbers $\mathbb{D}$

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- Vitali decomposition: $\mathbb{T}/\mathbb{D} = \{\{y \in \mathbb{T} : x - y \in \mathbb{D}\} : x \in \mathbb{T}\}$

\[
f : \omega 2/\text{Fin} \xrightarrow{1-1} \mathbb{T}/\mathbb{D}
\]

if there exists a selector for the Vitali decomposition, then a Vitali set is the set of cardinality $\aleph$.

Vitali set on the circle $\mathbb{T}$ for the set of all rational numbers $\mathbb{Q}$

\[
\mathbb{T}/\mathbb{Q} \cong (\mathbb{T}/\mathbb{D})/(\mathbb{Q}/\mathbb{D})
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A set $A \subseteq \mathbb{T}$ is called a **tail-set** if the set $\{r \in \mathbb{T} : A + r = A\}$ contains a countable subset dense in $\mathbb{T}$.

**Theorem 10** (J. Mycielski [1])

If $AC_2$ holds true, then there exist a Lebesgue non-measurable set of reals and a set which does not possess the Baire Property, i.e. $LM \rightarrow \neg AC_2$ and $BP \rightarrow \neg AC_2$.

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If a set of cardinality $k$ is linearly ordered, then there exist a Lebesgue non-measurable set of reals and a set which does not possess the Baire Property, i.e. $\text{LM} \rightarrow \neg \text{Lk}$ and $\text{BP} \rightarrow \neg \text{Lk}$.
A set $A \subseteq \mathbb{T}$ is called a **tail-set** if the set \( \{ r \in \mathbb{T} : A + r = A \} \) contains a countable subset dense in $\mathbb{T}$.

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Regularity properties on the real line

\[ \neg AC_2 \]

\[ \neg BS \]

\[ \neg VS \]

\[ \neg Lk \]

\[ \neg WR \]

\[ \neg Wk \]

**In1:** \( c < k \ll c \)

**In2:** \( \aleph_1 < \aleph_1 + c < \aleph_1 + k; \)

**BS:** there exists a Bernstein set

**WR:** the set of \( R \) can be well-ordered

**VS:** there exists a selector for a Vitali set

**LM:** every set of \( R \) is Lebesgue measurable

**BP:** every set of \( R \) possesses the Baire property

**Wk:** a set of cardinality \( k \) can be well-ordered

**Lk:** a set of cardinality \( k \) can be linearly ordered
A free ultrafilter on $\omega$ is a filter $\mathcal{I} \subseteq \mathcal{P}(\omega)$ not containing any finite set and for every $A \subseteq \omega$, either $A \in \mathcal{I}$ or $\omega \setminus A \in \mathcal{I}$.

Theorem 12
If the real line can be well-ordered, then there exists a free ultrafilter on $\omega$, i.e. $WR \rightarrow FU$.

Theorem 13 (W. Sierpiński [1])
A free ultrafilter on $\omega$ is a Lebesgue non-measurable set and does not possess the Baire Property, i.e. $LM \rightarrow \neg FU$ and $BP \rightarrow \neg FU$. 
A free ultrafilter on $\omega$ is a filter $\mathcal{J} \subseteq \mathcal{P}(\omega)$ not containing any finite set and for every $A \subseteq \omega$, either $A \in \mathcal{J}$ or $\omega \setminus A \in \mathcal{J}$.

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**Theorem 12**

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**Theorem 13 (W. Sierpiński [1])**

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Regularity properties on the real line

- AC₂: there exists a Bernstein set
- WR: the set of R can be well-ordered
- LM: every set of R is Lebesgue measurable
- BP: every set of R possesses the Baire property
- VS: there exists a selector for a Vitali set
- In₁: \( c < k \ll c \)
- In₂: \( \aleph_1 < \aleph_1 + c < \aleph_1 + k \)
- FU: there exists a free ultrafilter on \( \omega \)
- BS: there exists a Bernstein set
- Wk: a set of cardinality \( k \) can be well-ordered
- Lk: a set of cardinality \( k \) can be linearly ordered
- LM: every set of R is Lebesgue measurable
- BP: every set of R possesses the Baire property
- some kind of duality between measure and category, J. Raisonnier [3] proved in the theory $\text{ZF} + \text{wAC}$ that

**Theorem 14 (J. Raisonnier)**
If $\aleph_1 \leq \mathfrak{c}$, then there is a Lebesgue non-measurable set, i.e. $\text{LM} \rightarrow \text{Inc}$.

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**Theorem 14 (J. Raisonnier)**

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Regularity properties on the real line

**wAC**

LM: every set of R is Lebesgue measurable

BP: every set of R possesses the Baire property

¬AC₂: there exists a Bernstein set

WR: the set of R can be well-ordered

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¬VS: there exists a selector for a Vitali set

Lk: a set of cardinality k can be linearly ordered

Wk: a set of cardinality k can be well-ordered

¬FU: there exists a free ultrafilter on ω

Inc: ℵ₁ and c are incomparable

BS: there exists a Bernstein set

In1: c < k ≪ c

In2: ℵ₁ < ℵ₁ + c < ℵ₁ + k

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**Wk**: a set of cardinality k can be well-ordered

**Lk**: a set of cardinality k can be linearly ordered
Theorem 15

If \( w\text{CH} \) holds true, then the following are equivalent:

WR  the set of reals \( \mathbb{R} \) can be well-ordered;

Inc  \( \aleph_1 \) and \( c \) are comparable, i.e \( \aleph_1 \leq c \);

LDe  there exists a selector for the Lebesgue decomposition.

- If \( \aleph_1 \) and \( c \) are incomparable, then \( c = 2^{\aleph_0} < 2^{\aleph_1} \). Thus, we get Inc \( \rightarrow \) In3.
- From \( \aleph_1 < 2^{\aleph_1} \) we have \( w\text{CH} \rightarrow \text{In3} \).
Theorem 15

If $wCH$ holds true, then the following are equivalent:

- **WR** the set of reals $\mathbb{R}$ can be well-ordered;
- **Inc** $\aleph_1$ and $c$ are comparable, i.e. $\aleph_1 \leq c$;
- **LDe** there exists a selector for the Lebesgue decomposition.

- If $\aleph_1$ and $c$ are incomparable, then $c = 2^{\aleph_0} < 2^{\aleph_1}$. Thus, we get $Inc \rightarrow In3$.
- From $\aleph_1 < 2^{\aleph_1}$ we have $wCH \rightarrow In3$. 
Theorem 15

If $wCH$ holds true, then the following are equivalent:

1. **WR** the set of reals $\mathbb{R}$ can be well-ordered;
2. $\neg Inc$ $\aleph_1$ and $c$ are comparable, i.e $\aleph_1 \leq c$;
3. **LDe** there exists a selector for the Lebesgue decomposition.

- If $\aleph_1$ and $c$ are incomparable, then $c = 2^{\aleph_0} < 2^{\aleph_1}$. Thus, we get $Inc \rightarrow In3$.
- From $\aleph_1 < 2^{\aleph_1}$ we have $wCH \rightarrow In3$.
Theorem 15

If \( \text{wCH} \) holds true, then the following are equivalent:

- **WR** the set of reals \( \mathbb{R} \) can be well-ordered;
- \( \neg \text{Inc} \) \( \aleph_1 \) and \( c \) are comparable, i.e. \( \aleph_1 \leq c \);
- **LDe** there exists a selector for the Lebesgue decomposition.

- If \( \aleph_1 \) and \( c \) are incomparable, then \( c = 2^{\aleph_0} < 2^{\aleph_1} \). Thus, we get \( \text{Inc} \rightarrow \text{In3} \).
- From \( \aleph_1 < 2^{\aleph_1} \) we have \( \text{wCH} \rightarrow \text{In3} \).
Theorem 15

If \( wCH \) holds true, then the following are equivalent:

- **WR** the set of reals \( \mathbb{R} \) can be well-ordered;
- \( \neg Inc \) \( \aleph_1 \) and \( c \) are comparable, i.e. \( \aleph_1 \leq c \);
- **LDe** there exists a selector for the Lebesgue decomposition.

- If \( \aleph_1 \) and \( c \) are incomparable, then \( c = 2^{\aleph_0} < 2^{\aleph_1} \). Thus, we get \( Inc \rightarrow In3 \).
- From \( \aleph_1 < 2^{\aleph_1} \) we have \( wCH \rightarrow In3 \).
Theorem 15

If $wCH$ holds true, then the following are equivalent:

- **WR** the set of reals $\mathbb{R}$ can be well-ordered;
- $\neg$**Inc** $\aleph_1$ and $c$ are comparable, i.e $\aleph_1 \leq c$;
- **LDe** there exists a selector for the Lebesgue decomposition.

- If $\aleph_1$ and $c$ are incomparable, then $c = 2^{\aleph_0} < 2^{\aleph_1}$. Thus, we get $\text{Inc} \rightarrow \text{In3}$.
- From $\aleph_1 < 2^{\aleph_1}$ we have $wCH \rightarrow \text{In3}$.
Theorem 15

If $wCH$ holds true, then the following are equivalent:

- **WR** the set of reals $\mathbb{R}$ can be well-ordered;
- **$\neg Inc$** $\mathbb{N}_1$ and $c$ are comparable, i.e. $\mathbb{N}_1 \leq c$;
- **LDe** there exists a selector for the Lebesgue decomposition.

- If $\mathbb{N}_1$ and $c$ are incomparable, then $c = 2^{\mathbb{N}_0} < 2^{\mathbb{N}_1}$. Thus, we get $Inc \rightarrow In3$.
- From $\mathbb{N}_1 < 2^{\mathbb{N}_1}$ we have $wCH \rightarrow In3$. 

Regularity properties on the real line
Theorem 15

If $\textbf{wCH}$ holds true, then the following are equivalent:

WR the set of reals $\mathbb{R}$ can be well-ordered;

$\neg$Inc $\aleph_1$ and $\mathfrak{c}$ are comparable, i.e $\aleph_1 \leq \mathfrak{c}$;

LDe there exists a selector for the Lebesgue decomposition.

- If $\aleph_1$ and $\mathfrak{c}$ are incomparable, then $\mathfrak{c} = 2^{\aleph_0} < 2^{\aleph_1}$. Thus, we get $\textbf{Inc} \rightarrow \textbf{In3}$.
- from $\aleph_1 < 2^{\aleph_1}$ we have $\textbf{wCH} \rightarrow \textbf{In3}$
**LM**: every set of \( \mathbb{R} \) is Lebesgue measurable

**BP**: every set of \( \mathbb{R} \) possesses the Baire property

**¬BS**: there does not exist a Bernstein set

**In3**: \( c \neq 2^{\aleph_1} \)

**In1**: \( c < k \ll c \)

**¬WR**: the set of \( \mathbb{R} \) cannot be well-ordered

**¬AC**: there does not exist a selector for a Vitali set

**In2**: \( \aleph_1 < \aleph_1 + c < \aleph_1 + k \)

**Inc**: \( \aleph_1 \) and \( c \) are incomparable

**¬Lk**: a set of cardinality \( k \) cannot be linearly ordered

**¬Wk**: there does not exist a selection for a well-ordering

**¬FU**: there does not exist a free ultrafilter on \( \omega \)

**¬LDe**: there does not exist a selector for a Lebesgue decomposition

**wCH**: there is no set \( X \) such that \( \aleph_0 < |X| < c \)

**BP**: every set of \( \mathbb{R} \) possesses the Baire property

**LM**: every set of \( \mathbb{R} \) is Lebesgue measurable

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**In3**: \( c \neq 2^{\aleph_1} \)

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**wCH**: there is no set \( X \) such that \( \aleph_0 < |X| < c \)

**BP**: every set of \( \mathbb{R} \) possesses the Baire property
Regularity properties on the real line

- **LM**: every set of $\mathbb{R}$ is Lebesgue measurable
- **BP**: every set of $\mathbb{R}$ possesses the Baire property
- **VS**: there exists a selector for a Vitali set
- **In1**: $c < k \ll c$
- **In2**: $\aleph_1 < \aleph_1 + c < \aleph_1 + k$
- **In3**: $c \neq 2^{\aleph_1}$
- **Inc**: $\aleph_1$ and $c$ are incomparable
- **Wk**: the set of $\mathbb{R}$ can be well-ordered
- **WR**: the set of $\mathbb{R}$ can be well-ordered
- **AC**: there exists a Bernstein set
- **FU**: there exists a free ultrafilter on $\omega$
- **BS**: there exists a Bernstein set
- **Lk**: a set of cardinality $k$ can be linearly ordered
- **LDe**: there exists a selector for Lebesgue decomp.
- **Inc**: there is no set $X$ such that $\aleph_0 < |X| < c$
- **wCH**: there is no set $X$ such that $\aleph_0 < |X| < c$
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LM: every set of $\mathbb{R}$ is Lebesgue measurable
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\textbf{Regularity properties on the real line}

\textbf{LM}: every set of R is Lebesgue measurable
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\textbf{¬AC}: there exists a selector for a Vitali set
\textbf{In1}: $c < k \ll c$
\textbf{In2}: $\aleph_1 < \aleph_1 + c < \aleph_1 + k$
\textbf{In3}: $c \neq 2^\aleph_1$

\textbf{¬In1:} $\aleph_1 \notin \{k \mid k\}$

\textbf{¬In2:} $\aleph_1 \notin \{k \mid k\}$

\textbf{FU}: there exists a free ultrafilter on $\omega$

\textbf{LDe}: there exists a selector for Lebesgue decomp.

\textbf{Lk}: a set of cardinality $k$ can be linearly ordered

\textbf{¬Lk}: a set of cardinality $k$ can not be linearly ordered

\textbf{¬LDe}: there exists a selector for Lebesgue decomp.

\textbf{¬Inc}: there exists a selector for Lebesgue decomp.

\textbf{wCH}: there is no set $X$ such that $\aleph_0 < |X| < c$

\textbf{¬wCH}: there exists a set $X$ such that $\aleph_0 < |X| < c$

\textbf{CH}: $\aleph_1 = c$

\textbf{Inc}: $\aleph_1$ and $c$ are incomparable

\textbf{BS}: there exists a Bernstein set

\textbf{In1}: $c < k \ll c$

\textbf{In2}: $\aleph_1 < \aleph_1 + c < \aleph_1 + k$

\textbf{In3}: $c \neq 2^\aleph_1$
Theorem 16
If every uncountable set of reals contains a perfect subset, then there is no set $X$ such that $\aleph_0 < |X| < c$, i.e. $\text{PSP} \rightarrow \text{wCH}$.

Theorem 17
If every uncountable set of reals contains a perfect subset, then $\aleph_1$ and $c$ are incomparable, i.e. $\text{PSP} \rightarrow \text{Inc}$. 
Theorem 16
If every uncountable set of reals contains a perfect subset, then there is no set $X$ such that $\aleph_0 < |X| < c$, i.e. $\text{PSP} \rightarrow \text{wCH}$.

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If every uncountable set of reals contains a perfect subset, then $\aleph_1$ and $c$ are incomparable, i.e. $\text{PSP} \rightarrow \text{Inc}$.
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If every uncountable set of reals contains a perfect subset, then $\aleph_1$ and $c$ are incomparable, i.e. $\text{PSP} \rightarrow \text{Inc.}$.
Regularity properties on the real line

LM: every set of $\mathbb{R}$ is Lebesgue measurable
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¬BS: there exists a Bernstein set
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In1: $c < k \ll c$
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PSP: every uncount. set of $\mathbb{R}$ contains a perfect set

CH: $\aleph_1 = c$
In1: $c < k \ll c$
In3: $c \neq 2^\aleph_1$

LD: there exists a selector for Lebesgue decomp.
Lk: a set of cardinality $k$ can be linearly ordered
PSP: every uncount. set of $\mathbb{R}$ contains a perfect set
wCH: there is no set $X$ such that $\aleph_0 < |X| < c$
Negative implications:

- according to Theorem 15

\[ \text{wCH} \land \text{WR} \equiv \text{CH} \]

- by K. Gödel constructible universe \( L \) we have a model in which

\[ \text{wCH} \not\rightarrow \neg\text{WR}, \ \text{In3} \not\rightarrow \neg\text{WR}, \]

\[ \text{wCH} \not\leftrightarrow \text{Inc}, \ \text{In3} \not\leftrightarrow \text{Inc}, \]

\[ \text{wCH} \not\leftrightarrow \neg\text{LDe}, \ \text{In3} \not\leftrightarrow \neg\text{LDe}. \]
Negative implications:

- according to Theorem 15

\[ \text{wCH} \land \text{WR} \equiv \text{CH} \]

- by K. Gödel constructible universe \( L \) we have a model in which

\[ \text{wCH} \not\iff \lnot\text{WR}, \quad \text{In3} \not\iff \lnot\text{WR}, \]

\[ \text{wCH} \not\iff \text{Inc}, \quad \text{In3} \not\iff \text{Inc}, \]

\[ \text{wCH} \not\iff \lnot\text{LDe}, \quad \text{In3} \not\iff \lnot\text{LDe}. \]
Negative implications:

- according to Theorem 15

\[ \text{wCH} \land \text{WR} \equiv \text{CH} \]

- by K. Gödel constructible universe \( \mathbb{L} \) we have a model in which

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\[ \text{wCH} \not\leftrightarrow \text{Inc}, \ \text{In3} \not\leftrightarrow \text{Inc}, \]

\[ \text{wCH} \not\leftrightarrow \neg \text{LDe}, \ \text{In3} \not\leftrightarrow \neg \text{LDe}. \]
Negative implications:

- according to Theorem 15

\[ \text{wCH} \wedge \text{WR} \equiv \text{CH} \]

- by K. Gödel constructible universe L we have a model in which

\[ \text{wCH} \nleftrightarrow \neg \text{WR}, \ \text{In3} \nleftrightarrow \neg \text{WR}, \]

\[ \text{wCH} \nleftrightarrow \text{Inc}, \ \text{In3} \nleftrightarrow \text{Inc}, \]

\[ \text{wCH} \nleftrightarrow \neg \text{LDe}, \ \text{In3} \nleftrightarrow \neg \text{LDe}. \]
Negative implications:

- according to Theorem 15

\[ w\text{CH} \land WR \equiv CH \]

- by K. Gödel constructible universe \( L \) we have a model in which

\[ w\text{CH} \nrightarrow \neg WR, \ \text{In3} \nrightarrow \neg WR, \]

\[ w\text{CH} \nrightarrow \text{Inc}, \ \text{In3} \nrightarrow \text{Inc}, \]

\[ w\text{CH} \nrightarrow \neg \text{LDe}, \ \text{In3} \nrightarrow \neg \text{LDe}. \]
Negative implications:

- according to Theorem 15

\[ \text{wCH} \land \text{WR} \equiv \text{CH} \]

- by K. Gödel constructible universe \( L \) we have a model in which

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\[ \text{wCH} \nleftrightarrow \neg \text{LDe}, \quad \text{In3} \nleftrightarrow \neg \text{LDe}. \]
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- according to Theorem 15

\[ \text{wCH} \land \text{WR} \equiv \text{CH} \]

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\[ \text{wCH} \nleftrightarrow \neg \text{LDe}, \ \text{In3} \nleftrightarrow \neg \text{LDe}. \]
The **Axiom of Determinacy AD** states that every two-person games of length $\omega$ in which both players choose integers is determined; that is, one of the two players has a winning strategy.

- AD was proposed as an alternative to the Axiom of Choice by J. Mycielski and H. Steinhaus [2], but it is not possible to prove the consistency of $\text{ZF} + \text{AD}$ with respect to $\text{ZF}$,
- the consistency strength of AD is indicated as much high in due to results by Solovay and mainly by T. Jech [4].
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Theorem 18 (J. Mycielski, R. Solovay)

If $\mathbf{AD}$ holds true, then

a) $wAC$, $PSP$, $LM$, $BP$ hold true,

b) $AC$ fails,

c) there exists a surjection of $P(\omega)$ onto $P(\omega_1)$, i.e. $2^{\omega_1} < c = 2^{\omega_0}$. 
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\[ 2^{\aleph_1} \ll c = 2^{\aleph_0}. \]
Regularity properties on the real line

LM: every set of R is Lebesgue measurable
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AD: there exists a selector for Lebesgue decomp.
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By R. Solovay [2] and by S. Shelah [4] the following theories are equiconsistent

(a) $\text{ZFC + IC}$;\(^1\)
(b) $\text{ZFC + every } \Sigma^1_3\text{-set of reals is Lebesgue measurable}$;
(c) $\text{ZF + DC + LM}$.

\(^1\)IC denote statement “there exists a strongly inaccessible cardinal”, i.e. a limit regular cardinal $\kappa$ such that for any $\lambda < \kappa$ we have $2^\lambda < \kappa$. 

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Theorem 19

If $wAC$ holds true then $\aleph_1$ is a regular cardinal.

- by the Shelah’s argument in his Remark (1) of [4], the theory $ZF + wAC + LM$

is equiconsistent with the previous theories (a)-(c).
- S. Shelah proved that the consistency of $ZF$ implies the consistency of $ZF + wAC + BP$, i.e. the theories
  (d) $ZF$
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If \( w\text{AC} \) holds true then \( \aleph_1 \) is a regular cardinal.

- by the Shelah’s argument in his Remark (1) of [4], the theory \( \text{ZF} + w\text{AC} + \text{LM} \) is equiconsistent with the previous theories (a)-(c).
- S. Shelah proved that the consistency of \( \text{ZF} \) implies the consistency of \( \text{ZF} + w\text{AC} + \text{BP} \), i.e. the theories (d) \( \text{ZF} \) (e) \( \text{ZF} + w\text{AC} + \text{BP} \) are equiconsistent.
Theorem 19

If $\mathbf{wAC}$ holds true then $\aleph_1$ is a regular cardinal.

- by the Shelah’s argument in his Remark (1) of [4], the theory $\mathbf{ZF + wAC + LM}$ is equiconsistent with the previous theories (a)-(c).
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(d) $\mathbf{ZF}$

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by Shelah’s model the consistency strength of $\text{ZF} + \text{wAC} + \text{LM}$ is strictly greater than that of $\text{ZF} + \text{wAC} + \text{BP}$,

by Solovay’s model the consistency of $\text{ZF} + \text{wAC} + \text{LM}$ is greater than that of $\text{ZF} + \text{wAC} + \text{PSP}$.

Thus, a natural question arises:

**Question**

Is the consistency of the existence of an inaccessible cardinal necessary for PSP?

We give a positive answer to this question :)
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We give a positive answer to this question :)
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Thus, a natural question arises:

**Question**

Is consistency of the existence of an inaccessible cardinal necessary for $PSP$?

We give a positive answer to this question :)
by Shelah’s model the consistency strength of $\text{ZF} + \text{wAC} + \text{LM}$ is strictly greater than that of $\text{ZF} + \text{wAC} + \text{BP}$,

by Solovay’s model the consistency of $\text{ZF} + \text{wAC} + \text{LM}$ is greater than that of $\text{ZF} + \text{wAC} + \text{PSP}$.

Thus, a natural question arises:

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Is consistency of the existence of an inaccessible cardinal necessary for $\text{PSP}$?

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Theorem 20

If \( \text{PSP} \) holds true and \( \aleph_1 \) is a regular cardinal, then \( \aleph_1 \) is an inaccessible cardinal in the constructible universe \( L \).

- the theory \( \text{ZF} + \aleph_1 \text{ is regular} + \text{PSP} \) is equiconsistent with the theories (a)-(c)

Since the theories (d)-(e) are equiconsistent with the theory \( \text{ZF} + w\text{CH} \), we obtain

- the consistency of \( \text{ZF} + w\text{AC} + \text{PSP} \) is strictly greater than that of \( \text{ZF} + w\text{AC} + w\text{CH} \).

S. Shelah [4] showed that Theorem 14 on the Baire Property is not provable in the theory \( \text{ZF} + \text{DC} \).
Theorem 20

If $\text{PSP}$ holds true and $\aleph_1$ is a regular cardinal, then $\aleph_1$ is an inaccessible cardinal in the constructible universe $L$.

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Thus, we get:

- $BP \not\rightarrow Inc$,
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- however, according to Theorem 14 we have $BP \not\leftrightarrow LM$. 
Thus, we get:

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Regularity properties on the real line

Diagram in which none of the indicated implications is provable in the theory $\textbf{ZF} + \textbf{DC}$

$\neg\text{AC}_2$, Inc, In3

$\neg\text{AC}$

wAC

AD

LM

PSP

BP

$\neg\text{BS}$

$\neg\text{CH}$

$\neg\text{VS}$

$\neg\text{FU}$

$\neg\text{Lk}$

$\neg\text{LDe}$

$\neg\text{Wk}$

$\neg\text{WR}$

CH: $\aleph_1 = c$

In1: $c < k \ll c$

In3: $c \neq 2^{\aleph_1}$

In1

$\aleph_1 < \aleph_1 + c < \aleph_1 + k$;

Inc: $\aleph_1$ and $c$ are incomparable

BS: there exists a Bernstein set

FU: there exists a free ultrafilter on $\omega$

WR: the set of R can be well-ordered

VS: there exists a selector for a Vitali set

wCH: there is no set $X$ such that $\aleph_0 < |X| < c$

LDe: there exists a selector for Lebesgue decomp.

Lk: a set of cardinality $k$ can be linearly ordered

PSP: every uncount. set of R contains a perfect set

LM: every set of R is Lebesgue measurable

BP: every set of R possessest the Baire property
Theorem 21

If there is no selector for the Lebesgue decomposition and $\aleph_1$ is a regular cardinal, then $\aleph_1$ is an inaccessible cardinal in the constructible universe $L$.

Since $\aleph_1$ is not inaccessible in $L$ in the Shelah's above mentioned model, we obtain

- $\text{BP} \not\rightarrow \neg\text{LDe}$,
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A topological space $\langle X, \mathcal{O} \rangle$ is a Fréchet space iff \( \overline{A} = \text{scI}(A) = \{ \lim_{n \to \infty} x_n : (\forall n) x_n \in A \} \) for every set $A \subseteq X$.

Theorem (H. Herrlich)

wAC holds true if and only if the real line is a Fréchet space.

\( c < \kappa \rightarrow (\aleph_1, c \text{ are incomparable}) \lor (\aleph_1 < \aleph_1 + c < \aleph_1 + \kappa) \)

i.e. $\text{In1} \rightarrow \text{Inc} \lor \text{In2}$

J. Mycielski’s statement:

\( \neg Lk \rightarrow \text{In4}, \)

\( \text{In4} \rightarrow \aleph_1 < \aleph_1 + c < \aleph_1 + \kappa, \)

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**Theorem (H. Herrlich)**

\( w\text{AC} \) holds true if and only if the real line is a Fréchet space.

\[ c < \ell \rightarrow (\aleph_1, c \text{ are incomparable}) \lor (\aleph_1 < \aleph_1 + c < \aleph_1 + \ell) \]

i.e. \( \text{ln1} \rightarrow \text{lnC} \lor \text{ln2} \)

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References


References


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