Weak Reflection Principle, Saturation of NS and Diamonds

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Stationary sets in two cardinals version

Definition

We say that a set $S \subseteq [\lambda]^{\mu}$ is stationary if for every function $f : \lambda^{<\omega} \to \lambda$, there is $X \in S$ such that $f[X^{<\omega}] \subseteq X$.

Weak Reflection Principle

Definition ($\mathrm{WRP}(\lambda)$)

Let $\lambda \geq \aleph_2$ be an arbitrary ordinal. If $S \subseteq [\lambda]^\omega$ is a stationary set (in $[\lambda]^\omega$), then the set

$$\{x \in [\lambda]^{\omega_1} : x \supseteq \omega_1 \text{ and } S \cap [x]^\omega \text{ is stationary in } [x]^\omega\}$$

is stationary in $[\lambda]^{\omega_1}$. So WRP states that $\mathrm{WRP}(\lambda)$ holds for every $\lambda \geq \aleph_2$.

Saturation of NS_{ω_1}

Definition (Saturation of NS_{ω_1})

Let W be a collection of stationary sets in ω_1 such that for every S and T in W, $S \cap T$ is nonstationary. Then $|W| \leq \omega_1$.

Some consequences of WRP

1. WRP implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević 1984).
2. WRP implies SPFA is equivalent to MM (Foreman-Magidor-Shelah 1988).
3. WRP does not imply $\aleph_2^{\aleph_1} = \aleph_2$ (Woodin 1999).
4. WRP implies SCH (Shelah, 2008).

Diamond in two cardinals version

Definition

Let $\langle \mathscr{G}_Z \rangle_{Z \in [\lambda]^\mu}$ be a sequence such that $\mathscr{G}_Z \subseteq P(Z)$ and

$$|\mathscr{G}_Z| \leq \mu$$

for all $Z \in [\lambda]^\mu$. Then $\langle \mathscr{G}_Z \rangle_{Z \in [\lambda]^\mu}$ is a $\diamondsuit_{[\lambda]^\mu}$-sequence if for all $W \subseteq \lambda$, the set

$$\{Z \in [\lambda]^\mu : W \cap Z \in \mathscr{G}_Z\}$$

is stationary. The principle $\diamondsuit_{[\lambda]^\mu}$ states that there is a $\diamondsuit_{[\lambda]^\mu}$-sequence.

Main theorem

Theorem

For every ordinal $\lambda \geq \omega_2$, saturation of the ideal NS_{ω_1} and $\mathrm{WRP}(\lambda)$ imply $\diamondsuit_{[\lambda]^{\omega_1}}$.In particular, it implies $\diamondsuit_{\omega_2}(\{\delta < \omega_2 : \mathrm{cof}\ \delta = \omega_1\})$.

Building the sequence...

We will use the following theorem of Todorčević:

Theorem

$\diamondsuit_{[\lambda]^\omega}$ *holds for every ordinal* $\lambda \geq \omega_2$.

In fact, we shall also rely on the definition of $\diamondsuit_{[\lambda]^\omega}$.

Building the sequence...

Let $\langle \theta_a \rangle_{a \in [\lambda]^\omega}$ be a fixed a $\diamondsuit_{[\lambda]^\omega}$-sequence. We also fix for each $x \in [\lambda]^{\omega_1}$ a $\subseteq$-continuous increasing chain $\langle a_\xi^x \rangle_{\xi < \omega_1}$ of countable sets such that $x = \bigcup_{\xi < \omega_1} a_\xi^x$.

Let $T_x = \langle \{a_\xi^x\}_{\xi < \omega_1}, <_x \rangle$ be the associated tree where the ordering is as follows: $a_\xi^x <_x a_{\xi'}^x$ iff $\xi < \xi'$ and $\theta_{a_{\xi'}^x} \cap a_\xi^x = \theta_{a_\xi^x}$.

Building the sequence...

Take $S \subseteq \omega_1$ with the property that

$$\{a^x_\xi : \xi \in S\}$$

is a chain in the tree order.
For each S with this property, let

$$F^x_S = \bigcup_{\xi \in S} \theta_{a^x_\xi}.$$

Building the sequence...

Now, consider

$$\mathscr{S}_x = \{F_S^x : S \text{ is stationary in } \omega_1 \text{ and } \langle a_\xi^x \rangle_{\xi \in S} \text{ is a } <_x\text{-chain}\}.$$

Claim

Saturation of NS_{ω_1} *implies that* $|\mathscr{S}_x| \leq \omega_1$.

Claim

WRP implies $\langle \mathscr{S}_x \rangle_{x \in [\lambda]^{\omega_1}}$ *is a* $\diamondsuit_{[\lambda]^{\omega_1}}$*-sequence. In particular, it implies* $\diamondsuit_{\omega_2}(\{\delta < \omega_2 : \mathrm{cof}\ \delta = \omega_1\})$.