Moron Maps and subspaces of N^* under PFA

Alan Dow

Department of Mathematics
University of North Carolina Charlotte

winter school 2010

using the tricks to study autohomeomorphisms

Fix a function $\Phi : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N})$ which is a *lifting* of a mod fin homomorphism: (dually $\mathbb{N}^* \overset{f}{\leftarrow} \mathbb{N}^*$, $\Phi(X) =^* f^{-1}(X^*)$)

$$\Phi(X) \cup \Phi(Y) =^* \Phi(X \cup Y);\ \ \Phi(X) \cap \Phi(Y) =^* \Phi(X \cap Y);\ \ \Phi(\emptyset) = \emptyset$$

using the tricks to study autohomeomorphisms

Fix a function $\Phi : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N})$ which is a *lifting* of a mod fin homomorphism: (dually $\mathbb{N}^* \overset{f}{\leftarrow} \mathbb{N}^*$, $\Phi(X) =^* f^{-1}(X^*)$)

$$\Phi(X) \cup \Phi(Y) =^* \Phi(X \cup Y); \;\; \Phi(X) \cap \Phi(Y) =^* \Phi(X \cap Y); \;\; \Phi(\emptyset) = \emptyset$$

as in the OCA discussion, $\mathcal{P}(\mathbb{N})$ is given the topology with $[s; n] = \{X \subset \mathbb{N} : X \cap n = s\}$ being the basic clopen sets

using the tricks to study autohomeomorphisms

Fix a function $\Phi : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N})$ which is a *lifting* of a mod fin homomorphism: (dually $\mathbb{N}^* \overset{f}{\leftarrow} \mathbb{N}^*$, $\Phi(X) =^* f^{-1}(X^*)$)

$$\Phi(X) \cup \Phi(Y) =^* \Phi(X \cup Y); \;\; \Phi(X) \cap \Phi(Y) =^* \Phi(X \cap Y); \;\; \Phi(\emptyset) = \emptyset$$

as in the OCA discussion, $\mathcal{P}(\mathbb{N})$ is given the topology with $[s; n] = \{X \subset \mathbb{N} : X \cap n = s\}$ being the basic clopen sets

say that $I \in triv(\Phi)$ if there is an $h_I \in \mathbb{N}^I$ such that $\Phi(a) =^* h_I[a]$ for all $a \subset I$

using the tricks to study autohomeomorphisms

Fix a function $\Phi : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N})$ which is a *lifting* of a mod fin homomorphism: (dually $\mathbb{N}^* \overset{f}{\leftarrow} \mathbb{N}^*$, $\Phi(X) =^* f^{-1}(X^*)$)

$$\Phi(X) \cup \Phi(Y) =^* \Phi(X \cup Y); \ \ \Phi(X) \cap \Phi(Y) =^* \Phi(X \cap Y); \ \ \Phi(\emptyset) = \emptyset$$

as in the OCA discussion, $\mathcal{P}(\mathbb{N})$ is given the topology with $[s; n] = \{X \subset \mathbb{N} : X \cap n = s\}$ being the basic clopen sets

say that $I \in triv(\Phi)$ if there is an $h_I \in \mathbb{N}^I$ such that $\Phi(a) =^* h_I[a]$ for all $a \subset I$

note that h_I is continuous as a map on $\mathcal{P}(I)$

using the tricks to study autohomeomorphisms

Fix a function $\Phi : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N})$ which is a *lifting* of a mod fin homomorphism: (dually $\mathbb{N}^* \overset{f}{\leftarrow} \mathbb{N}^*$, $\Phi(X) =^* f^{-1}(X^*)$)

$$\Phi(X) \cup \Phi(Y) =^* \Phi(X \cup Y); \;\; \Phi(X) \cap \Phi(Y) =^* \Phi(X \cap Y); \;\; \Phi(\emptyset) = \emptyset$$

as in the OCA discussion, $\mathcal{P}(\mathbb{N})$ is given the topology with $[s; n] = \{X \subset \mathbb{N} : X \cap n = s\}$ being the basic clopen sets

say that $I \in triv(\Phi)$ if there is an $h_I \in \mathbb{N}^I$ such that $\Phi(a) =^* h_I[a]$ for all $a \subset I$

note that h_I is continuous as a map on $\mathcal{P}(I)$

if there is a continuous lifting then Φ is trivial.

adding a Cohen real to a Borel lifting F

adding a Cohen real to a Borel lifting F

Suppose F is a Borel lifting of automorphism Φ (continuous on dense G_δ $\mathcal{X}$)

adding a Cohen real to a Borel lifting F

Suppose F is a Borel lifting of automorphism Φ (continuous on dense G_δ $\mathcal{X}$)

Let g_1, g_2 be generics for the poset $\{[s; n] : s \subset n \in \mathbb{N}\}$

adding a Cohen real to a Borel lifting F

Suppose F is a Borel lifting of automorphism Φ (continuous on dense G_δ $\mathcal{X}$)

Let g_1, g_2 be generics for the poset $\{[s; n] : s \subset n \in \mathbb{N}\}$

Then $F(g_1)$ and $F(g_2)$ are defined, but $\Phi(g_i)$ are not;

$\exists [s_1; n], [s_2; n] \Vdash (F(g_1) \star F(g_2)) \Delta F(g_1 \star g_2) \subset n$
where $\star$ is one of $\{\cap, \cup, \Delta, -\}$

adding a Cohen real to a Borel lifting F

Suppose F is a Borel lifting of automorphism Φ (continuous on dense G_δ $\mathcal{X}$)

Let g_1, g_2 be generics for the poset $\{[s;n] : s \subset n \in \mathbb{N}\}$

Then $F(g_1)$ and $F(g_2)$ are defined, but $\Phi(g_i)$ are not;

$\exists [s_1; n], [s_2; n] \Vdash (F(g_1) \star F(g_2)) \Delta F(g_1 \star g_2) \subset n$
where $\star$ is one of $\{\cap, \cup, \Delta, -\}$
also $(\exists \tilde{n} > n)\ \forall s, t \subset n$ (and extend s_1, s_2 with $s_1 \Delta s_2 \subset n$)
$[s_1; n], [s_2; n] \Vdash F(s \cup g_1 - n) - \tilde{n} = F(t \cup g_1 - n) - \tilde{n}$

adding a Cohen real to a Borel lifting F

Suppose F is a Borel lifting of automorphism Φ (continuous on dense G_δ $\mathcal{X}$)

Let g_1, g_2 be generics for the poset $\{[s; n] : s \subset n \in \mathbb{N}\}$

Then $F(g_1)$ and $F(g_2)$ are defined, but $\Phi(g_i)$ are not;

$\exists [s_1; n], [s_2; n] \Vdash (F(g_1) \star F(g_2)) \Delta F(g_1 \star g_2) \subset n$
where $\star$ is one of $\{\cap, \cup, \Delta, -\}$
also $(\exists \tilde{n} > n)$ $\forall s, t \subset n$ (and extend s_1, s_2 with $s_1 \Delta s_2 \subset n$)
$[s_1; n], [s_2; n] \Vdash F(s \cup g_1 - n) - \tilde{n} = F(t \cup g_1 - n) - \tilde{n}$

otherwise meeting countably many dense sets, including some to get inside dense G_δ set $\mathcal{X}$, we find $v_1, v_2 \subset \mathbb{N}$ yielding, e.g.
$\Phi(v_1) \star \Phi(v_2) =^* F(v_1) \star F(v_2) \neq^* F(v_1 \star v_2) =^* \Phi(v_1 \star v_2)$

completely additive implies trivial

Now in the extension: by continuity for $a \in \mathcal{X}$

$\Phi(a) =^* F_1(a) = \lim_m F_1((a \cap m) \cup g_i - m)$ and
$F_1(a) = \lim_m F_1((a \cap m) \cup (g_1 \star g_2) - m)$

completely additive implies trivial

Now in the extension: by continuity for $a \in \mathcal{X}$

$\Phi(a) =^* F_1(a) = \lim_m F_1((a \cap m) \cup g_i - m)$ and
$F_1(a) = \lim_m F_1((a \cap m) \cup (g_1 \star g_2) - m)$

and for $a, b \in \mathcal{X}$
$\Phi(a \star b) =^* \Phi(a) \star \Phi(b) =^* F_1(a) \star F_1(b) = F_1(a \star b)$

completely additive implies trivial

Now in the extension: by continuity for $a \in \mathcal{X}$

$\Phi(a) =^* F_1(a) = \lim_m F_1((a \cap m) \cup g_i - m)$ and
$F_1(a) = \lim_m F_1((a \cap m) \cup (g_1 \star g_2) - m)$

and for $a, b \in \mathcal{X}$
$\Phi(a \star b) =^* \Phi(a) \star \Phi(b) =^* F_1(a) \star F_1(b) = F_1(a \star b)$

little bit easy exercise, $(\forall x \subset \mathbb{N})$, there are $a, b \in \mathcal{X}$ such that $x = a \Delta b$,

hence F_1 has a unique continuous extension, $\tilde{F}$, to $\mathcal{P}(\mathbb{N})$, and this is a *pure* lifting

completely additive implies trivial

Now in the extension: by continuity for $a \in \mathcal{X}$

$\Phi(a) =^* F_1(a) = \lim_m F_1((a \cap m) \cup g_i - m)$ and
$F_1(a) = \lim_m F_1((a \cap m) \cup (g_1 \star g_2) - m)$

and for $a, b \in \mathcal{X}$
$\Phi(a \star b) =^* \Phi(a) \star \Phi(b) =^* F_1(a) \star F_1(b) = F_1(a \star b)$

little bit easy exercise, $(\forall x \subset \mathbb{N})$, there are $a, b \in \mathcal{X}$ such that $x = a \Delta b$,

hence F_1 has a unique continuous extension, $\tilde{F}$, to $\mathcal{P}(\mathbb{N})$, and this is a *pure* lifting

completely additive implies trivial

Now in the extension: by continuity for $a \in \mathcal{X}$

$\Phi(a) =^* F_1(a) = \lim_m F_1((a \cap m) \cup g_i - m)$ and
$F_1(a) = \lim_m F_1((a \cap m) \cup (g_1 \star g_2) - m)$

and for $a, b \in \mathcal{X}$
$\Phi(a \star b) =^* \Phi(a) \star \Phi(b) =^* F_1(a) \star F_1(b) = F_1(a \star b)$

little bit easy exercise, $(\forall x \subset \mathbb{N})$, there are $a, b \in \mathcal{X}$ such that $x = a \Delta b$,

hence F_1 has a unique continuous extension, $\tilde{F}$, to $\mathcal{P}(\mathbb{N})$, and this is a *pure* lifting

now define $h(i) \in \tilde{F}(\{i\})$ for $a \in \mathcal{X}$ and check that h induces Φ

Cohen forcing and σ-Borel automorphisms

Theorem: let Φ be a lifting of a mod fin homomorphism which has no Borel lifting, then adding a Cohen real will not add a continuous lifting for $\Phi \restriction V \cap \mathcal{P}(\mathbb{N})$.

Cohen forcing and σ-Borel automorphisms

Theorem: let Φ be a lifting of a mod fin homomorphism which has no Borel lifting, then adding a Cohen real will not add a continuous lifting for $\Phi \restriction V \cap \mathcal{P}(\mathbb{N})$.

(skipping) proof: Assume that $F : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathbb{N})$ is a continuous function (after forcing with $2^{<\omega}$) and that $F(X) =^* \Phi(X)$ for all $X \in \mathcal{P}(\mathbb{N})$.
Put $X \in \mathbb{X}_{p,n}$ providing $p \Vdash F(X) \setminus n = \Phi(X) \setminus n$.
Find p, n and $s \subset n$ such that $\mathbb{X}_{p,n}$ is dense in $[s; n]$

Let $Y \in [s; n] \cap V$ and let $\{X_k : k \in \omega\} \subset \mathbb{X}_{p,n} \cap [s; n]$ converge to Y. Then $p \Vdash F(Y) = \lim_k F(X_k) =^* \Phi(Y)$, hence $F(Y) \in V$.

Thus, $\Phi_s(X) = \Phi(s \cup (X \setminus n))$ is a continuous lifting for the same homomorphism.

Cohen forcing and σ-Borel automorphisms

Theorem: let Φ be a lifting of a mod fin homomorphism which has no Borel lifting, then adding a Cohen real will not add a continuous lifting for $\Phi \restriction V \cap \mathcal{P}(\mathbb{N})$.

Cohen forcing and σ-Borel automorphisms

Theorem: let Φ be a lifting of a mod fin homomorphism which has no Borel lifting, then adding a Cohen real will not add a continuous lifting for $\Phi \restriction V \cap \mathcal{P}(\mathbb{N})$.

more Borel map and Cohen connection:

Cohen forcing and σ-Borel automorphisms

Theorem: let Φ be a lifting of a mod fin homomorphism which has no Borel lifting, then adding a Cohen real will not add a continuous lifting for $\Phi \restriction V \cap \mathcal{P}(\mathbb{N})$.

more Borel map and Cohen connection:

If $\dot{Y}$ is a Cohen (i.e. $P = \{[s; n] : s \subset n \in \mathbb{N}\}$) name of $\subset \mathbb{N}$, then there is a Borel map (continuous on a dense G_δ) $F_{\dot{Y}}$ such that, in the extension, $F_{\dot{Y}}(g) = val_g(\dot{Y})$

Cohen forcing and σ-Borel automorphisms

Theorem: let Φ be a lifting of a mod fin homomorphism which has no Borel lifting, then adding a Cohen real will not add a continuous lifting for $\Phi \restriction V \cap \mathcal{P}(\mathbb{N})$.

more Borel map and Cohen connection:

If $\dot{Y}$ is a Cohen (i.e. $P = \{[s; n] : s \subset n \in \mathbb{N}\}$) name of $\subset \mathbb{N}$, then there is a Borel map (continuous on a dense G_δ) $F_{\dot{Y}}$ such that, in the extension, $F_{\dot{Y}}(g) = val_g(\dot{Y})$

AND, **Lemma** there are $x \subset a \subset \mathbb{N}$, $\mathbb{N} \setminus a \notin \text{triv}(\Phi)$ such that $\Vdash F_{\dot{Y}}(x \cup (g \setminus a)) \cap \Phi(a) \neq^* \Phi(x)$

i.e. $\Vdash_{P_{x,a}} \dot{Y} \cap \Phi(a) \neq^* \Phi(x)$

Velickovic also proved σ-Borel implies Borel

there is quite a tricky step to this theorem which seems to simplify if we again throw Cohen forcing at it.

Velickovic also proved σ-Borel implies Borel

there is quite a tricky step to this theorem which seems to simplify if we again throw Cohen forcing at it.

Velickovic also proved σ-Borel implies Borel

there is quite a tricky step to this theorem which seems to simplify if we again throw Cohen forcing at it.

Assume that $\{F_n : n \in \omega\}$ is a family of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for all $X \subset \mathbb{N}$, there is an n such that $\Phi(X) = F_n(X)$.

Velickovic also proved σ-Borel implies Borel

there is quite a tricky step to this theorem which seems to simplify if we again throw Cohen forcing at it.

Assume that $\{F_n : n \in \omega\}$ is a family of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for all $X \subset \mathbb{N}$, there is an n such that $\Phi(X) = F_n(X)$.

Apply above Lemma to obtain $x_0 \subset a_0 \subset \mathbb{N}$ with $\mathbb{N} \setminus a_0 \notin triv(\Phi)$,

and $\Vdash F_0(x_0 \cup (g \setminus a_0)) \cap \Phi(a_0) \neq^* \Phi(x_0)$

Velickovic also proved σ-Borel implies Borel

there is quite a tricky step to this theorem which seems to simplify if we again throw Cohen forcing at it.

Assume that $\{F_n : n \in \omega\}$ is a family of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for all $X \subset \mathbb{N}$, there is an n such that $\Phi(X) = F_n(X)$.

Apply above Lemma to obtain $x_0 \subset a_0 \subset \mathbb{N}$ with $\mathbb{N} \setminus a_0 \notin triv(\Phi)$,

and $\Vdash F_0(x_0 \cup (g \setminus a_0)) \cap \Phi(a_0) \neq^* \Phi(x_0)$

this hands us countably many dense sets that we must *protect*

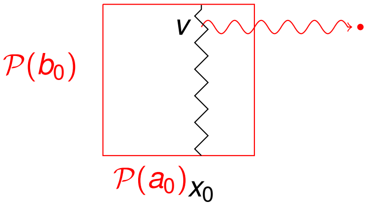
σ-Borel, Borel, continuous are all the same

σ-Borel, Borel, continuous are all the same

Claim: (there is) x_0 such that for comeager many $v \subset b_0$, $F_0(x_0 \cup v) \cap \Phi(a_0) \neq^* \Phi(x_0)$.

σ-Borel, Borel, continuous are all the same

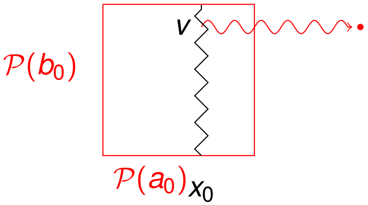
Claim: (there is) x_0 such that for comeager many $v \subset b_0$, $F_0(x_0 \cup v) \cap \Phi(a_0) \neq^* \Phi(x_0)$.



$= F_0(x_0 \cup v) \cap \Phi(a_0)$

σ-Borel, Borel, continuous are all the same

Claim: (there is) x_0 such that for comeager many $v \subset b_0$, $F_0(x_0 \cup v) \cap \Phi(a_0) \neq^* \Phi(x_0)$.

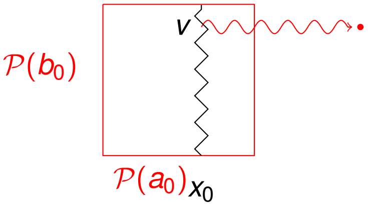


$= F_0(x_0 \cup v) \cap \Phi(a_0)$

otherwise, let g be $\mathcal{P}(b_0)$-generic, and set $H(x) = F_0(x \cup g) \cap \Phi(a_0)$ and we have Cohen added a continuous lifting for $\Phi \restriction \mathcal{P}(a_0)$

σ-Borel, Borel, continuous are all the same

Claim: (there is) x_0 such that for comeager many $v \subset b_0$, $F_0(x_0 \cup v) \cap \Phi(a_0) \neq^* \Phi(x_0)$.



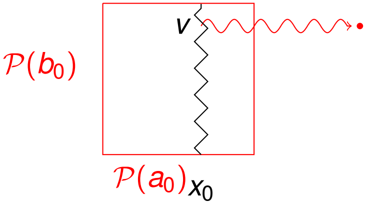
$= F_0(x_0 \cup v) \cap \Phi(a_0)$

otherwise, let g be $\mathcal{P}(b_0)$-generic, and set $H(x) = F_0(x \cup g) \cap \Phi(a_0)$ and we have Cohen added a continuous lifting for $\Phi \restriction \mathcal{P}(a_0)$

repeat this, obtaining $x_k \subset a_k \subset b_{k-1}$ so that $\Phi(x_k) \neq^* F_k(x_0 \cup \cdots x_k \cup v) \cap \Phi(a_k)$ for comeager many $v \subset b_k = b_{k-1} \setminus a_k$.

σ-Borel, Borel, continuous are all the same

Claim: (there is) x_0 such that for comeager many $v \subset b_0$, $F_0(x_0 \cup v) \cap \Phi(a_0) \neq^* \Phi(x_0)$.



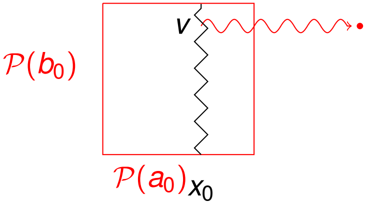
$= F_0(x_0 \cup v) \cap \Phi(a_0)$

otherwise, let g be $\mathcal{P}(b_0)$-generic, and set $H(x) = F_0(x \cup g) \cap \Phi(a_0)$ and we have Cohen added a continuous lifting for $\Phi \restriction \mathcal{P}(a_0)$

repeat this, obtaining $x_k \subset a_k \subset b_{k-1}$ so that $\Phi(x_k) \neq^* F_k(x_0 \cup \cdots x_k \cup v) \cap \Phi(a_k)$ for comeager many $v \subset b_k = b_{k-1} \setminus a_k$. Also carefully ensure that $v = \bigcup_{j>k} x_j$ lands you in the appropriate comeager sets.

σ-Borel, Borel, continuous are all the same

Claim: (there is) x_0 such that for comeager many $v \subset b_0$, $F_0(x_0 \cup v) \cap \Phi(a_0) \neq^* \Phi(x_0)$.



$= F_0(x_0 \cup v) \cap \Phi(a_0)$

otherwise, let g be $\mathcal{P}(b_0)$-generic, and set $H(x) = F_0(x \cup g) \cap \Phi(a_0)$ and we have Cohen added a continuous lifting for $\Phi \restriction \mathcal{P}(a_0)$

repeat this, obtaining $x_k \subset a_k \subset b_{k-1}$ so that $\Phi(x_k) \neq^* F_k(x_0 \cup \cdots x_k \cup v) \cap \Phi(a_k)$ for comeager many $v \subset b_k = b_{k-1} \setminus a_k$. Also carefully ensure that $v = \bigcup_{j>k} x_j$ lands you in the appropriate comeager sets. Then $\Phi(v) \neq^* F_k(v)$ for all k.

Shelah-Steprans Q and A; Step 1

More Cohen forcing connections.

Shelah-Steprans Q and A; Step 1

More Cohen forcing connections.

Lemma 1 again: Let $\dot{Y}$ be a $P = [\mathbb{N}]^{<\omega}$-name of a subset of $\mathbb{N}$ and Φ be a homomorphism. Let $\dot{Y} \in M \prec H(\theta)$ be countable and let $\mathfrak{D}$ be the set of dense subsets of P. Let b not in $triv(\Phi)$.

Shelah-Steprans Q and A; Step 1

More Cohen forcing connections.
Lemma 1 again: Let $\dot{Y}$ be a $P = [\mathbb{N}]^{<\omega}$-name of a subset of $\mathbb{N}$ and Φ be a homomorphism. Let $\dot{Y} \in M \prec H(\theta)$ be countable and let $\mathfrak{D}$ be the set of dense subsets of P. Let b not in $triv(\Phi)$.

For $x \subset a \subset b$, $P_{x,a} = \{p \in P : p \cap a = x \cap \max(p)\}$.

Shelah-Steprans Q and A; Step 1

More Cohen forcing connections.
Lemma 1 again: Let $\dot{Y}$ be a $P = [\mathbb{N}]^{<\omega}$-name of a subset of $\mathbb{N}$ and Φ be a homomorphism. Let $\dot{Y} \in M \prec H(\theta)$ be countable and let $\mathfrak{D}$ be the set of dense subsets of P. Let b not in $triv(\Phi)$.

For $x \subset a \subset b$, $P_{x,a} = \{p \in P : p \cap a = x \cap \max(p)\}$. We assume b is one of the "many" sets b so that for all $x \subset a \subset b$, $D \cap P_{x,a}$ is dense in $P_{x,a}$ for all $D \in \mathfrak{D} \cap M$.

Shelah-Steprans Q and A; Step 1

More Cohen forcing connections.
Lemma 1 again: Let $\dot{Y}$ be a $P = [\mathbb{N}]^{<\omega}$-name of a subset of $\mathbb{N}$ and Φ be a homomorphism. Let $\dot{Y} \in M \prec H(\theta)$ be countable and let $\mathfrak{D}$ be the set of dense subsets of P. Let b not in $triv(\Phi)$.

For $x \subset a \subset b$, $P_{x,a} = \{p \in P : p \cap a = x \cap \max(p)\}$. We assume b is one of the "many" sets b so that for all $x \subset a \subset b$, $D \cap P_{x,a}$ is dense in $P_{x,a}$ for all $D \in \mathfrak{D} \cap M$.

If g is P-generic, then $g_{x,a} = x \cup (g \setminus a)$ is generic for $P_{x,a}$.

Shelah-Steprans Q and A; Step 1

More Cohen forcing connections.
Lemma 1 again: Let $\dot{Y}$ be a $P = [\mathbb{N}]^{<\omega}$-name of a subset of $\mathbb{N}$ and Φ be a homomorphism. Let $\dot{Y} \in M \prec H(\theta)$ be countable and let $\mathfrak{D}$ be the set of dense subsets of P. Let b not in $triv(\Phi)$.

For $x \subset a \subset b$, $P_{x,a} = \{p \in P : p \cap a = x \cap \max(p)\}$. We assume b is one of the "many" sets b so that for all $x \subset a \subset b$, $D \cap P_{x,a}$ is dense in $P_{x,a}$ for all $D \in \mathfrak{D} \cap M$.

If g is P-generic, then $g_{x,a} = x \cup (g \setminus a)$ is generic for $P_{x,a}$.

In $V[g]$, we know that $F_{\dot{Y}}$ is Borel on $\mathcal{P}(a)$. and that there are $x \subset a \subset b$ with $b \setminus a \notin triv(\Phi)$

Shelah-Steprans Q and A; Step 1

More Cohen forcing connections.

Lemma 1 again: Let $\dot{Y}$ be a $P = [\mathbb{N}]^{<\omega}$-name of a subset of $\mathbb{N}$ and Φ be a homomorphism. Let $\dot{Y} \in M \prec H(\theta)$ be countable and let $\mathfrak{D}$ be the set of dense subsets of P. Let b not in $triv(\Phi)$.

For $x \subset a \subset b$, $P_{x,a} = \{p \in P : p \cap a = x \cap \max(p)\}$. We assume b is one of the "many" sets b so that for all $x \subset a \subset b$, $D \cap P_{x,a}$ is dense in $P_{x,a}$ for all $D \in \mathfrak{D} \cap M$.

If g is P-generic, then $g_{x,a} = x \cup (g \setminus a)$ is generic for $P_{x,a}$.

In $V[g]$, we know that $F_{\dot{Y}}$ is Borel on $\mathcal{P}(a)$. and that there are $x \subset a \subset b$ with $b \setminus a \notin triv(\Phi)$

such that $1 \Vdash_{P_{x,a}} \Phi(x) \neq^* \dot{Y}_{g_{x,a}} \cap \Phi(a)$.

Shelah-Steprans Q and A: Step 2 – ω_1

Shelah-Steprans Q and A: Step 2 – ω_1

recursively construct $\mathcal{A} = \{a_\xi \supset x_\xi : \xi \in \omega_1\}$ as above, so that $\mathcal{A} = \{a_\xi : \xi \in \omega_1\}$ is mod fin increasing and $x_\alpha \cap a_\xi =^* x_\xi$ for $\xi < \alpha$, and $\mathbb{N} \setminus a_\alpha \notin triv(\Phi)$

Shelah-Steprans Q and A: Step 2 – ω_1

recursively construct $\mathcal{A} = \{a_\xi \supset x_\xi : \xi \in \omega_1\}$ as above, so that $\mathcal{A} = \{a_\xi : \xi \in \omega_1\}$ is mod fin increasing and $x_\alpha \cap a_\xi =^* x_\xi$ for $\xi < \alpha$, and $\mathbb{N} \setminus a_\alpha \notin triv(\Phi)$

Define $Q_\alpha \subset [\mathbb{N}]^{<\omega} \times [\alpha]^{<\omega}$ by

$(q, J) \in Q_\alpha$ implies $(x_\xi \cap a_\eta)\Delta x_\eta \subset \max q$ for $\xi < \eta \in J$

Shelah-Steprans Q and A: Step 2 $-\ \omega_1$

recursively construct $\mathcal{A} = \{a_\xi \supset x_\xi : \xi \in \omega_1\}$ as above, so that $\mathcal{A} = \{a_\xi : \xi \in \omega_1\}$ is mod fin increasing and $x_\alpha \cap a_\xi =^* x_\xi$ for $\xi < \alpha$, and $\mathbb{N} \setminus a_\alpha \notin triv(\Phi)$
Define $Q_\alpha \subset [\mathbb{N}]^{<\omega} \times [\alpha]^{<\omega}$ by

$(q, J) \in Q_\alpha$ implies $(x_\xi \cap a_\eta)\Delta x_\eta \subset \max q$ for $\xi < \eta \in J$

and ordered (so as to mimic P_{x_ξ, a_ξ} for all $\xi < \alpha$)

Shelah-Steprans Q and A: Step 2 $-\ \omega_1$

recursively construct $\mathcal{A} = \{a_\xi \supset x_\xi : \xi \in \omega_1\}$ as above, so that $\mathcal{A} = \{a_\xi : \xi \in \omega_1\}$ is mod fin increasing and $x_\alpha \cap a_\xi =^* x_\xi$ for $\xi < \alpha$, and $\mathbb{N} \setminus a_\alpha \notin triv(\Phi)$

Define $Q_\alpha \subset [\mathbb{N}]^{<\omega} \times [\alpha]^{<\omega}$ by

$(q, J) \in Q_\alpha$ implies $(x_\xi \cap a_\eta)\Delta x_\eta \subset \max q$ for $\xi < \eta \in J$

and ordered (so as to mimic P_{x_ξ, a_ξ} for all $\xi < \alpha$)

$(p, H) < (q, J)$ implies

$(p \setminus \max(q)) \cap a_\xi = x_\xi \cap (\max(q), \max(p)]$ for all $\xi \in J$.

Shelah-Steprans Q and A: Step 2 – ω_1

recursively construct $\mathcal{A} = \{a_\xi \supset x_\xi : \xi \in \omega_1\}$ as above, so that $\mathcal{A} = \{a_\xi : \xi \in \omega_1\}$ is mod fin increasing and $x_\alpha \cap a_\xi =^* x_\xi$ for $\xi < \alpha$, and $\mathbb{N} \setminus a_\alpha \notin triv(\Phi)$

Define $Q_\alpha \subset [\mathbb{N}]^{<\omega} \times [\alpha]^{<\omega}$ by

$(q, J) \in Q_\alpha$ implies $(x_\xi \cap a_\eta)\Delta x_\eta \subset \max q$ for $\xi < \eta \in J$

and ordered (so as to mimic P_{x_ξ, a_ξ} for all $\xi < \alpha$)

$(p, H) < (q, J)$ implies

$(p \setminus \max(q)) \cap a_\xi = x_\xi \cap (\max(q), \max(p)]$ for all $\xi \in J$.

Simultaneously be selecting an increasing chain ($\diamondsuit$) of countable elementary submodels so that $Q_\alpha \in M_\alpha$ and the choice of a_α ensures all dense $D \subset Q_\alpha$ from M_α remain dense in $Q_{\alpha+1}$. (at limits there's a P-ideal issue)

Shelah-Steprans Q and A: Step 2 – ω_1

recursively construct $\mathcal{A} = \{a_\xi \supset x_\xi : \xi \in \omega_1\}$ as above, so that $\mathcal{A} = \{a_\xi : \xi \in \omega_1\}$ is mod fin increasing and $x_\alpha \cap a_\xi =^* x_\xi$ for $\xi < \alpha$, and $\mathbb{N} \setminus a_\alpha \notin triv(\Phi)$

Define $Q_\alpha \subset [\mathbb{N}]^{<\omega} \times [\alpha]^{<\omega}$ by

$(q, J) \in Q_\alpha$ implies $(x_\xi \cap a_\eta)\Delta x_\eta \subset \max q$ for $\xi < \eta \in J$

and ordered (so as to mimic P_{x_ξ, a_ξ} for all $\xi < \alpha$)

$(p, H) < (q, J)$ implies

$(p \setminus \max(q)) \cap a_\xi = x_\xi \cap (\max(q), \max(p)]$ for all $\xi \in J$.

Simultaneously be selecting an increasing chain ($\diamondsuit$) of countable elementary submodels so that $Q_\alpha \in M_\alpha$ and the choice of a_α ensures all dense $D \subset Q_\alpha$ from M_α remain dense in $Q_{\alpha+1}$. (at limits there's a P-ideal issue)

Also ensure that for all $\dot{Y} \in M_\alpha$ which are Q_α-names, $\dot{Y} g_{x_\alpha, a_\alpha} \cap \Phi(a_\alpha) \neq^* \Phi(x_\alpha)$.

what does this get us?

One of the roles the models M_α is to ensure that Q_{ω_1} (or $Q_{\mathcal{A}}$) is ccc (easy-ish $\diamondsuit$ argument).

what does this get us?

One of the roles the models M_α is to ensure that Q_{ω_1} (or $Q_{\mathcal{A}}$) is ccc (easy-ish $\diamondsuit$ argument).

Thus if $\dot{Y}$ is a nice Q_{ω_1}-name of a subset of $\mathbb{N}$, there is an α such that $\dot{Y} \in M_\alpha$.

what does this get us?

One of the roles the models M_α is to ensure that Q_{ω_1} (or $Q_{\mathcal{A}}$) is ccc (easy-ish $\diamondsuit$ argument).

Thus if $\dot{Y}$ is a nice Q_{ω_1}-name of a subset of $\mathbb{N}$, there is an α such that $\dot{Y} \in M_\alpha$.

Let $G \subset Q_{\omega_1}$ be generic and let $g = \bigcup\{p : \exists H \ \ (p, H) \in G\}$. For each $\alpha \in \omega_1$, there is a p_α so that $(p_\alpha, \{\alpha\}) \in G$, and with $p_\alpha \subset n_\alpha$ we have that $(g \cap a_\alpha)\Delta x_\alpha \subset n_\alpha$.

what does this get us?

One of the roles the models M_α is to ensure that Q_{ω_1} (or $Q_{\mathcal{A}}$) is ccc (easy-ish $\diamondsuit$ argument).

Thus if $\dot{Y}$ is a nice Q_{ω_1}-name of a subset of $\mathbb{N}$, there is an α such that $\dot{Y} \in M_\alpha$.

Let $G \subset Q_{\omega_1}$ be generic and let $g = \bigcup\{p : \exists H \ (p, H) \in G\}$. For each $\alpha \in \omega_1$, there is a p_α so that $(p_\alpha, \{\alpha\}) \in G$, and with $p_\alpha \subset n_\alpha$ we have that $(g \cap a_\alpha)\Delta x_\alpha \subset n_\alpha$.

The choice of $a_\alpha \supset x_\alpha$ and the fact that $g =^* x_\alpha \cup (g \setminus a_\alpha)$ is sufficiently $(Q_\alpha)_{x_\alpha, a_\alpha}$-generic, we have that $\dot{Y}_g \cap \Phi(a_\alpha) \neq^* \Phi(x_\alpha)$.

what does this get us?

One of the roles the models M_α is to ensure that Q_{ω_1} (or $Q_{\mathcal{A}}$) is ccc (easy-ish $\diamondsuit$ argument).

Thus if $\dot{Y}$ is a nice Q_{ω_1}-name of a subset of $\mathbb{N}$, there is an α such that $\dot{Y} \in M_\alpha$.

Let $G \subset Q_{\omega_1}$ be generic and let $g = \bigcup\{p : \exists H \ (p, H) \in G\}$. For each $\alpha \in \omega_1$, there is a p_α so that $(p_\alpha, \{\alpha\}) \in G$, and with $p_\alpha \subset n_\alpha$ we have that $(g \cap a_\alpha)\Delta x_\alpha \subset n_\alpha$.

The choice of $a_\alpha \supset x_\alpha$ and the fact that $g =^* x_\alpha \cup (g \setminus a_\alpha)$ is sufficiently $(Q_\alpha)_{x_\alpha, a_\alpha}$-generic, we have that $\dot{Y}_g \cap \Phi(a_\alpha) \neq^* \Phi(x_\alpha)$.

All this adds up to $\{c_\alpha = \Phi(x_\alpha), d_\alpha = \Phi(a_\alpha \setminus x_\alpha) : \alpha \in \omega_1\}$ is a *freezable* gap (while $\{x_\alpha, (a_\alpha \setminus x_\alpha)\}$ is split by g).

I know this is taking a while

pulling it all back via PFA, there is a sequence $\{a_\alpha \supset x_\alpha : \alpha \in \omega_1\}$ and a set $X \subset \mathbb{N}$ such that $X \cap a_\alpha =^* x_\alpha$ for all α, but, there is no set Y such that $Y \cap F(a_\alpha) =^* F(x_\alpha)$.

I know this is taking a while

pulling it all back via PFA, there is a sequence $\{a_\alpha \supset x_\alpha : \alpha \in \omega_1\}$ and a set $X \subset \mathbb{N}$ such that $X \cap a_\alpha =^* x_\alpha$ for all α, but, there is no set Y such that $Y \cap F(a_\alpha) =^* F(x_\alpha)$.

so it must fail. which means $triv(\Phi) \neq \emptyset$

I know this is taking a while

pulling it all back via PFA, there is a sequence $\{a_\alpha \supset x_\alpha : \alpha \in \omega_1\}$ and a set $X \subset \mathbb{N}$ such that $X \cap a_\alpha =^* x_\alpha$ for all α, but, there is no set Y such that $Y \cap F(a_\alpha) =^* F(x_\alpha)$.

so it must fail. which means $triv(\Phi) \neq \emptyset$

in any event,

I know this is taking a while

pulling it all back via PFA, there is a sequence $\{a_\alpha \supset x_\alpha : \alpha \in \omega_1\}$ and a set $X \subset \mathbb{N}$ such that $X \cap a_\alpha =^* x_\alpha$ for all α, but, there is no set Y such that $Y \cap F(a_\alpha) =^* F(x_\alpha)$.

so it must fail. which means $\mathit{triv}(\Phi) \neq \emptyset$

in any event,

Theorem: if we are stuck in choosing $a_\alpha \supset x_\alpha$, it is because $\mathit{triv}(\Phi)$ is ccc over fin and $\Vdash_{Q_\alpha} \Phi \restriction \{a_\beta : \beta < \alpha\}^\perp \cap V = \mathcal{J}$ is σ-Borel. (hence trivial on each $b \in \mathcal{J}$)

I know this is taking a while

pulling it all back via PFA, there is a sequence $\{a_\alpha \supset x_\alpha : \alpha \in \omega_1\}$ and a set $X \subset \mathbb{N}$ such that $X \cap a_\alpha =^* x_\alpha$ for all α, but, there is no set Y such that $Y \cap F(a_\alpha) =^* F(x_\alpha)$.

so it must fail. which means $triv(\Phi) \neq \emptyset$

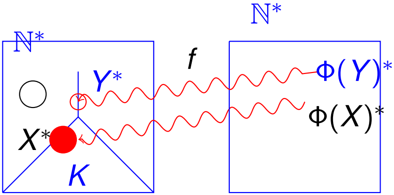
in any event,
Theorem: if we are stuck in choosing $a_\alpha \supset x_\alpha$, it is because $triv(\Phi)$ is ccc over fin and $\Vdash_{Q_\alpha} \Phi \restriction \{a_\beta : \beta < \alpha\}^\perp \cap V = \mathcal{J}$ is σ-Borel. (hence trivial on each $b \in \mathcal{J}$)

[Velickovic] σ-Borel plus trivial on each member of a P-ideal $\mathcal{J}$ implies there is a single h inducing Φ on each member of $\mathcal{J}$.

non-trivial embeddings of $\mathbb{N}^*$

now we'd like to note the important theorem of [Farah] PFA implies that if $K \subset \mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$, then the interior of K is clopen ($= A^*$) and $K \setminus A^*$ is ccc over fin.

Let f be a homeomorphism from $\mathbb{N}^*$ to K. Define the dual homomorphism Φ by $\Phi(X) \subset \mathbb{N}$ is such that $\Phi(X)^* = f^{-1}(X^* \cap K)$.



Since $X^* \cap \partial K \neq \emptyset$ means that $X \notin triv(\Phi)$, we have that ∂K is ccc over fin which shows that int(K) is clopen (i.e. regular closed sets do not have ccc over fin boundary)

some known consequences of trivial on ccc over fin

$\mathbb{N}^*$ does not map onto $(\omega \times \omega + 1)^*$ (let alone not being homeomorphic)

some known consequences of trivial on ccc over fin

$\mathbb{N}^*$ does not map onto $(\omega \times \omega + 1)^*$ (let alone not being homeomorphic)

the measure algebra does not embed in $\mathcal{P}(\mathbb{N})/\mathit{fin}$

some known consequences of trivial on ccc over fin

$\mathbb{N}^*$ does not map onto $(\omega \times \omega + 1)^*$ (let alone not being homeomorphic)

the measure algebra does not embed in $\mathcal{P}(\mathbb{N})/\mathit{fin}$

still open: if $\mathbb{N}^*$ maps onto compact ED X, must X be separable?

some known consequences of trivial on ccc over fin

$\mathbb{N}^*$ does not map onto $(\omega \times \omega + 1)^*$ (let alone not being homeomorphic)

the measure algebra does not embed in $\mathcal{P}(\mathbb{N})/\mathit{fin}$

still open: if $\mathbb{N}^*$ maps onto compact ED X, must X be separable?

$\mathbb{R}^*$ does not map onto the separable continuum: the Stone-Cech compactification of the long repeating topologist's sine curve (the closure of the graph of $\sin(1/(x - \lfloor x \rfloor))$)

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K.

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

By MA we are able to show that K is nowhere ccc and that for each open $U \subset K$, there is a $b \subset \mathbb{N}$ such that $f \restriction b^*$ is 1-to-1 into U.

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

By MA we are able to show that K is nowhere ccc and that for each open $U \subset K$, there is a $b \subset \mathbb{N}$ such that $f \restriction b^*$ is 1-to-1 into U.

Each G_δ contained in $f[a^*] \cap f[(\mathbb{N} \setminus a)^*]$ is contained in interior.

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

By MA we are able to show that K is nowhere ccc and that for each open $U \subset K$, there is a $b \subset \mathbb{N}$ such that $f \restriction b^*$ is 1-to-1 into U.

Each G_δ contained in $f[a^*] \cap f[(\mathbb{N} \setminus a)^*]$ is contained in interior.

Proof:

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

By MA we are able to show that K is nowhere ccc and that for each open $U \subset K$, there is a $b \subset \mathbb{N}$ such that $f \restriction b^*$ is 1-to-1 into U.

Each G_δ contained in $f[a^*] \cap f[(\mathbb{N} \setminus a)^*]$ is contained in interior.

Proof: Let $\bigcap_n U_n$ be the G_δ. May assume $\overline{U_{n+1}} \subset U_n$.

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

By MA we are able to show that K is nowhere ccc and that for each open $U \subset K$, there is a $b \subset \mathbb{N}$ such that $f \restriction b^*$ is 1-to-1 into U.

Each G_δ contained in $f[a^*] \cap f[(\mathbb{N} \setminus a)^*]$ is contained in interior.

Proof: Let $\bigcap_n U_n$ be the G_δ. May assume $\overline{U_{n+1}} \subset U_n$. Choose $z_n \in U_n \setminus f[a^*]$ (wlog) for all n.

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

By MA we are able to show that K is nowhere ccc and that for each open $U \subset K$, there is a $b \subset \mathbb{N}$ such that $f \restriction b^*$ is 1-to-1 into U.

Each G_δ contained in $f[a^*] \cap f[(\mathbb{N} \setminus a)^*]$ is contained in interior.

Proof: Let $\bigcap_n U_n$ be the G_δ. May assume $\overline{U_{n+1}} \subset U_n$. Choose $z_n \in U_n \setminus f[a^*]$ (wlog) for all n. Thus $f^{-1}(\{z_n\}_n)$ is a discrete set and its closure maps 2-to-1 onto the closure of $\{z_n\}_n$.

PFA resolves the 2-to-1 mapping question

Let f be a 2-to-1 map from $\mathbb{N}^*$ onto a space K. Let $\mathcal{I}$ be those $I \subset \mathbb{N}$ such that $f \restriction I^*$ is locally 1-to-1 and $f[I^*]$ open in K.

By MA we are able to show that K is nowhere ccc and that for each open $U \subset K$, there is a $b \subset \mathbb{N}$ such that $f \restriction b^*$ is 1-to-1 into U.

Each G_δ contained in $f[a^*] \cap f[(\mathbb{N} \setminus a)^*]$ is contained in interior.

Proof: Let $\bigcap_n U_n$ be the G_δ. May assume $\overline{U_{n+1}} \subset U_n$. Choose $z_n \in U_n \setminus f[a^*]$ (wlog) for all n. Thus $f^{-1}(\{z_n\}_n)$ is a discrete set and its closure maps 2-to-1 onto the closure of $\{z_n\}_n$. It follows that the limit points of $\{z_n\}$ while in $\bigcap_n U_n$ are not in $f[a^*]$.

clopen copies of $\mathbb{N}^*$ in K

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

We use the Shelah-Steprans Q and A method.

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

We use the Shelah-Steprans Q and A method.

Given any $U \subset K$, we can choose disjoint family $\{U_\alpha : \alpha \in \omega_1\}$ of open subsets of U.

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

We use the Shelah-Steprans Q and A method.

Given any $U \subset K$, we can choose disjoint family $\{U_\alpha : \alpha \in \omega_1\}$ of open subsets of U. For each α choose $W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$.

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

We use the Shelah-Steprans Q and A method.

Given any $U \subset K$, we can choose disjoint family $\{U_\alpha : \alpha \in \omega_1\}$ of open subsets of U. For each α choose $W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$.

Next find $a_\alpha \subset \mathbb{N}$ such that $f^{-1}(\overline{W_\alpha}) \subset a_\alpha^*$ and $a_\alpha^* \cap f^{-1}(K \setminus U_\alpha)$ is empty.

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

We use the Shelah-Steprans Q and A method.

Given any $U \subset K$, we can choose disjoint family $\{U_\alpha : \alpha \in \omega_1\}$ of open subsets of U. For each α choose $W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$.

Next find $a_\alpha \subset \mathbb{N}$ such that $f^{-1}(\overline{W_\alpha}) \subset a_\alpha^*$ and $a_\alpha^* \cap f^{-1}(K \setminus U_\alpha)$ is empty. Thus $f[(\mathbb{N} \setminus a_\alpha)^*]$ is disjoint from W_α.

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

We use the Shelah-Steprans Q and A method.

Given any $U \subset K$, we can choose disjoint family $\{U_\alpha : \alpha \in \omega_1\}$ of open subsets of U. For each α choose $W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$.

Next find $a_\alpha \subset \mathbb{N}$ such that $f^{-1}(\overline{W_\alpha}) \subset a_\alpha^*$ and $a_\alpha^* \cap f^{-1}(K \setminus U_\alpha)$ is empty. Thus $f[(\mathbb{N} \setminus a_\alpha)^*]$ is disjoint from W_α.

Fix any infinite $b_\alpha \subset a_\alpha$ such that $f[b_\alpha^*] \subset W_\alpha$ and $f \restriction b_\alpha^*$ is 1-to-1.

clopen copies of $\mathbb{N}^*$ in K

Lemma 2: for each open $U \subset K$, there is an $I \in \mathcal{I}$ such that $f[I^*] \subset U$. hence $\mathcal{I}$ is non-empty.

We use the Shelah-Steprans Q and A method.

Given any $U \subset K$, we can choose disjoint family $\{U_\alpha : \alpha \in \omega_1\}$ of open subsets of U. For each α choose $W_\alpha \subset \overline{W_\alpha} \subset U_\alpha$.

Next find $a_\alpha \subset \mathbb{N}$ such that $f^{-1}(\overline{W_\alpha}) \subset a_\alpha^*$ and $a_\alpha^* \cap f^{-1}(K \setminus U_\alpha)$ is empty. Thus $f[(\mathbb{N} \setminus a_\alpha)^*]$ is disjoint from W_α.

Fix any infinite $b_\alpha \subset a_\alpha$ such that $f[b_\alpha^*] \subset W_\alpha$ and $f \restriction b_\alpha^*$ is 1-to-1. If $f[b_\alpha^*]$ has any interior, we have succeeded. So, we assume instead, that for all α, $f[b_\alpha^*]$ is nowhere dense.

Before continuing, we ask if there is some such selection for which there is a set $A \subset \mathbb{N}$ such that $A \cap b_\alpha =^* \emptyset$ and $c_\alpha = A \cap a_\alpha$ still satisfies that $f[c_\alpha^*] \supset f[b_\alpha^*]$.

Before continuing, we ask if there is some such selection for which there is a set $A \subset \mathbb{N}$ such that $A \cap b_\alpha =^* \emptyset$ and $c_\alpha = A \cap a_\alpha$ still satisfies that $f[c_\alpha^*] \supset f[b_\alpha^*]$. If so, make this selection instead.

Before continuing, we ask if there is some such selection for which there is a set $A \subset \mathbb{N}$ such that $A \cap b_\alpha =^* \emptyset$ and $c_\alpha = A \cap a_\alpha$ still satisfies that $f[c_\alpha^*] \supset f[b_\alpha^*]$. If so, make this selection instead.

For each α then, there is a closed set $K_\alpha \subset c_\alpha^* = (a_\alpha \setminus b_\alpha)^*$ such that $f[K_\alpha] = f[b_\alpha^*]$

Before continuing, we ask if there is some such selection for which there is a set $A \subset \mathbb{N}$ such that $A \cap b_\alpha =^* \emptyset$ and $c_\alpha = A \cap a_\alpha$ still satisfies that $f[c_\alpha^*] \supset f[b_\alpha^*]$. If so, make this selection instead.

For each α then, there is a closed set $K_\alpha \subset c_\alpha^* = (a_\alpha \setminus b_\alpha)^*$ such that $f[K_\alpha] = f[b_\alpha^*]$

i.e. there is a homomorphism H_α from $\mathcal{P}(c_\alpha)$ onto $\mathcal{P}(b_\alpha)$.

Before continuing, we ask if there is some such selection for which there is a set $A \subset \mathbb{N}$ such that $A \cap b_\alpha =^* \emptyset$ and $c_\alpha = A \cap a_\alpha$ still satisfies that $f[c_\alpha^*] \supset f[b_\alpha^*]$. If so, make this selection instead.

For each α then, there is a closed set $K_\alpha \subset c_\alpha^* = (a_\alpha \setminus b_\alpha)^*$ such that $f[K_\alpha] = f[b_\alpha^*]$

i.e. there is a homomorphism H_α from $\mathcal{P}(c_\alpha)$ onto $\mathcal{P}(b_\alpha)$.

Following the Shelah-Steprans method, we can force with ${}^{<\omega_1}2$ and then construct a sequence $\{c_\alpha, d_\alpha : \alpha \in \omega_1\}$, so that the poset Q_{ω_1} is ccc and we obtain a gap from $\{H_\alpha(d_\alpha), H_\alpha(c_\alpha \setminus d_\alpha) : \alpha \in \omega_1\}$.

This gives us a set X (forced by Q_{ω_1}) satisfying that $X \cap c_\alpha =^* d_\alpha$ for all α. We are sure that there are uncountably many α such that $X \cap b_\alpha$ is not mod finite equal to $H_\alpha(d_\alpha)$.

This gives us a set X (forced by Q_{ω_1}) satisfying that $X \cap c_\alpha =^* d_\alpha$ for all α. We are sure that there are uncountably many α such that $X \cap b_\alpha$ is not mod finite equal to $H_\alpha(d_\alpha)$. By symmetry, we may assume that $\tilde{b}_\alpha = H_\alpha(d_\alpha) \setminus X$ is infinite.

This gives us a set X (forced by Q_{ω_1}) satisfying that $X \cap c_\alpha =^* d_\alpha$ for all α. We are sure that there are uncountably many α such that $X \cap b_\alpha$ is not mod finite equal to $H_\alpha(d_\alpha)$. By symmetry, we may assume that $\tilde{b}_\alpha = H_\alpha(d_\alpha) \setminus X$ is infinite. By the definition of H_α, it follows that $f[d_\alpha^*] \supset f[\tilde{b}_\alpha^*]$ and that X does separate the family of such d_α's and the $\tilde{b}_\alpha$'s.

This gives us a set X (forced by Q_{ω_1}) satisfying that $X \cap c_\alpha =^* d_\alpha$ for all α. We are sure that there are uncountably many α such that $X \cap b_\alpha$ is not mod finite equal to $H_\alpha(d_\alpha)$. By symmetry, we may assume that $\tilde{b}_\alpha = H_\alpha(d_\alpha) \setminus X$ is infinite. By the definition of H_α, it follows that $f[d^*_\alpha] \supset f[\tilde{b}^*_\alpha]$ and that X does separate the family of such d_α's and the $\tilde{b}_\alpha$'s. This means that we were able to choose A as above, and that we may assume that $X \subset A$.

This gives us a set X (forced by Q_{ω_1}) satisfying that $X \cap c_\alpha =^* d_\alpha$ for all α. We are sure that there are uncountably many α such that $X \cap b_\alpha$ is not mod finite equal to $H_\alpha(d_\alpha)$. By symmetry, we may assume that $\tilde{b}_\alpha = H_\alpha(d_\alpha) \setminus X$ is infinite. By the definition of H_α, it follows that $f[d^*_\alpha] \supset f[\tilde{b}^*_\alpha]$ and that X does separate the family of such d_α's and the $\tilde{b}_\alpha$'s. This means that we were able to choose A as above, and that we may assume that $X \subset A$.

We have the gap $\{H_\alpha(d_\alpha), b_\alpha \setminus H_\alpha(d_\alpha) : \alpha \in \omega_1\}$, which implies there is a point w in $\overline{\bigcup_\alpha (H_\alpha(d_\alpha))^*} \cap \overline{\bigcup_\alpha (b_\alpha \setminus H_\alpha(d_\alpha))^*} \subset (\mathbb{N} \setminus A)^*$

This gives us a set X (forced by Q_{ω_1}) satisfying that $X \cap c_\alpha =^* d_\alpha$ for all α. We are sure that there are uncountably many α such that $X \cap b_\alpha$ is not mod finite equal to $H_\alpha(d_\alpha)$. By symmetry, we may assume that $\tilde{b}_\alpha = H_\alpha(d_\alpha) \setminus X$ is infinite. By the definition of H_α, it follows that $f[d_\alpha^*] \supset f[\tilde{b}_\alpha^*]$ and that X does separate the family of such d_α's and the $\tilde{b}_\alpha$'s. This means that we were able to choose A as above, and that we may assume that $X \subset A$.

We have the gap $\{H_\alpha(d_\alpha), b_\alpha \setminus H_\alpha(d_\alpha) : \alpha \in \omega_1\}$, which implies there is a point w in $\overline{\bigcup_\alpha (H_\alpha(d_\alpha))^*} \cap \overline{\bigcup_\alpha (b_\alpha \setminus H_\alpha(d_\alpha))^*} \subset (\mathbb{N} \setminus A)^*$

$$f[(A \cap X)^*] \supset f[\overline{\textstyle\bigcup_\alpha (H_\alpha(d_\alpha))^*}] \text{ and}$$
$$f[(A \setminus X)^*] \supset f[\overline{\textstyle\bigcup_\alpha (b_\alpha \setminus H_\alpha(d_\alpha))^*}].$$

This gives us a set X (forced by Q_{ω_1}) satisfying that $X \cap c_\alpha =^* d_\alpha$ for all α. We are sure that there are uncountably many α such that $X \cap b_\alpha$ is not mod finite equal to $H_\alpha(d_\alpha)$. By symmetry, we may assume that $\tilde{b}_\alpha = H_\alpha(d_\alpha) \setminus X$ is infinite. By the definition of H_α, it follows that $f[d_\alpha^*] \supset f[\tilde{b}_\alpha^*]$ and that X does separate the family of such d_α's and the $\tilde{b}_\alpha$'s. This means that we were able to choose A as above, and that we may assume that $X \subset A$.

We have the gap $\{H_\alpha(d_\alpha), b_\alpha \setminus H_\alpha(d_\alpha) : \alpha \in \omega_1\}$, which implies there is a point w in $\overline{\bigcup_\alpha (H_\alpha(d_\alpha))^*} \cap \overline{\bigcup_\alpha (b_\alpha \setminus H_\alpha(d_\alpha))^*} \subset (\mathbb{N} \setminus A)^*$

$$f[(A \cap X)^*] \supset f[\overline{\bigcup_\alpha (H_\alpha(d_\alpha))^*}] \text{ and}$$
$$f[(A \setminus X)^*] \supset f[\overline{\bigcup_\alpha (b_\alpha \setminus H_\alpha(d_\alpha))^*}].$$

That means $f(w)$ has 3 points in its preimage!

non-empty G_δ's have non-empty interior

Next Lemma: K has the property that non-empty G_δ's have non-empty interior. (uses Farah's theorem)

Let $\{U_n\}_n$ be the sequence of open sets such that $\overline{U_{n+1}} \subset U_n$. For each n, we have some $(b_n \cup c_n) \in \mathcal{I}$ such that $f[b_n^*] = f[c_n^*] \subset U_n \setminus \overline{U_{n+1}}$ and is a clopen subset of K.

For each n, $f^{-1}(U_n)$ is an open set in $\mathbb{N}^*$ which contains the closure of $\bigcup_{k\geq n}(b_k \cup c_k)^*$. Thus we can arrange that $\left(\bigcup_{k\geq n}(b_k \cup c_k)\right)^*$ is contained in $f^{-1}(U_n)$ for each n.

If $U = K \setminus f[(\mathbb{N} \setminus \bigcup_n b_n)^*] \subset \bigcap_n U_n$ is not empty then we are done.

o/w, set $b = \bigcup_n b_n$ and notice that $f \restriction b^*$ must be 1-to-1 (since $f[(\mathbb{N} \setminus b)^*] \supset f[b^*]$).

By Farah's theorem, the canonical embedding given by $f^{-1} \circ f$ from b^* into $(\mathbb{N} \setminus b)^*$ will have the form $a^* \cup S$ where S is some nowhere dense set. Since c_n^* is contained in this image for each n, it follows that $c_n \subset^* a$ for each n. Choose any infinite $c \subset a$ such that $c \cap c_n$ is finite for each n. It follows that there is a $\tilde{b} \subset b$ such that $f[\tilde{b}^*] = f[c^*] \subset \bigcap_n U_n$ and again we have demonstrated that $\bigcap_n U_n$ contains an open set.

By Farah's theorem, the canonical embedding given by $f^{-1} \circ f$ from b^* into $(\mathbb{N} \setminus b)^*$ will have the form $a^* \cup S$ where S is some nowhere dense set. Since c_n^* is contained in this image for each n, it follows that $c_n \subset^* a$ for each n. Choose any infinite $c \subset a$ such that $c \cap c_n$ is finite for each n. It follows that there is a $\tilde{b} \subset b$ such that $f[\tilde{b}^*] = f[c^*] \subset \bigcap_n U_n$ and again we have demonstrated that $\bigcap_n U_n$ contains an open set.

By Farah's theorem, the canonical embedding given by $f^{-1} \circ f$ from b^* into $(\mathbb{N} \setminus b)^*$ will have the form $a^* \cup S$ where S is some nowhere dense set. Since c_n^* is contained in this image for each n, it follows that $c_n \subset^* a$ for each n. Choose any infinite $c \subset a$ such that $c \cap c_n$ is finite for each n. It follows that there is a $\tilde{b} \subset b$ such that $f[\tilde{b}^*] = f[c^*] \subset \bigcap_n U_n$ and again we have demonstrated that $\bigcap_n U_n$ contains an open set.

Then we use the CH * Cohen * OCA trick to finish as follows.

By Farah's theorem, the canonical embedding given by $f^{-1} \circ f$ from b^* into $(\mathbb{N} \setminus b)^*$ will have the form $a^* \cup S$ where S is some nowhere dense set. Since c_n^* is contained in this image for each n, it follows that $c_n \subset^* a$ for each n. Choose any infinite $c \subset a$ such that $c \cap c_n$ is finite for each n. It follows that there is a $\tilde{b} \subset b$ such that $f[\tilde{b}^*] = f[c^*] \subset \bigcap_n U_n$ and again we have demonstrated that $\bigcap_n U_n$ contains an open set.

Then we use the CH * Cohen * OCA trick to finish as follows.

Let $x \in \mathbb{N}^*$ be any point witnessing that f is not locally 1-to-1.

By Farah's theorem, the canonical embedding given by $f^{-1} \circ f$ from b^* into $(\mathbb{N} \setminus b)^*$ will have the form $a^* \cup S$ where S is some nowhere dense set. Since c_n^* is contained in this image for each n, it follows that $c_n \subset^* a$ for each n. Choose any infinite $c \subset a$ such that $c \cap c_n$ is finite for each n. It follows that there is a $\tilde{b} \subset b$ such that $f[\tilde{b}^*] = f[c^*] \subset \bigcap_n U_n$ and again we have demonstrated that $\bigcap_n U_n$ contains an open set.

Then we use the CH * Cohen * OCA trick to finish as follows.

Let $x \in \mathbb{N}^*$ be any point witnessing that f is not locally 1-to-1.

To save time, just assert that using non-empty G_δ's have non-empty interior in K, we can construct a sequence $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ converging to x

Probably skip the construction of $\{a_\alpha : \alpha \in \omega_1\}$

Fix any $E \in x$ such that $f(x) \in f[(\mathbb{N} \setminus E)^*]$. If there were any G_δ of K containing $f(x)$ and contained in $f[E^*] \cap f[(\mathbb{N} \setminus E)^*]$, then f would be locally 1-to-1 at x.

Suppose we are given any countable $\mathcal{A} \subset x$, we may by enlarging $\mathcal{A}$ assume that for each $a \in \mathcal{A}$, there is an $\tilde{a} \in \mathcal{A}$ such that $f[\tilde{a}^*] \cap f[(E \setminus a)^*]$ is empty.
$K \setminus \bigcup_{a \in \mathcal{A}} f[(E \setminus a)^*]$ is a G_δ containing $f(x)$ and so can not be contained in $f[(\mathbb{N} \setminus E)^*]$.
And since it has dense interior, there is a $b \in \mathcal{I}$ such that $f[b^*] \subset U$. It is easily checked that $b \prec \mathcal{A}$.

This completes the proof that given countable $\mathcal{A}$ from x, there is a $b \prec \mathcal{A}$ such that $b \in \mathcal{I}$.

now we finish the proof

now we finish the proof

Give the sequence $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ converging to x,

now we finish the proof

Give the sequence $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ converging to x, there are mappings $h_\alpha : a_\alpha \mapsto a_\alpha$ such that $h_\alpha^2 = id$ and $h_\alpha(n) \neq n$. (i.e. h_α induces $f^{-1} \circ f$), then for all uncountable $I \subset \omega_1$, $\bigcup_{\alpha \in I} h_\alpha$ is not 1-to-1 on any member of x.

now we finish the proof

Give the sequence $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ converging to x, there are mappings $h_\alpha : a_\alpha \mapsto a_\alpha$ such that $h_\alpha^2 = id$ and $h_\alpha(n) \neq n$. (i.e. h_α induces $f^{-1} \circ f$), then for all uncountable $I \subset \omega_1$, $\bigcup_{\alpha \in I} h_\alpha$ is not 1-to-1 on any member of x.

Force with $3^{<\mathbb{N}}$ thus adding a partition C_0, C_1, C_2

now we finish the proof

Give the sequence $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ converging to x, there are mappings $h_\alpha : a_\alpha \mapsto a_\alpha$ such that $h_\alpha^2 = id$ and $h_\alpha(n) \neq n$. (i.e. h_α induces $f^{-1} \circ f$), then for all uncountable $I \subset \omega_1$, $\bigcup_{\alpha \in I} h_\alpha$ is not 1-to-1 on any member of x.

Force with $3^{<\mathbb{N}}$ thus adding a partition C_0, C_1, C_2
$(\alpha, \beta) \in R$ (per OCA) if there are $i \in C_0 \cap a_\alpha$, $j \in C_1 \cap a_\beta$ so that $h_\alpha(i) = h_\beta(j) \in C_2$.

now we finish the proof

Give the sequence $\{a_\alpha : \alpha \in \omega_1\} \subset \mathcal{I}$ converging to x, there are mappings $h_\alpha : a_\alpha \mapsto a_\alpha$ such that $h_\alpha^2 = id$ and $h_\alpha(n) \neq n$. (i.e. h_α induces $f^{-1} \circ f$), then for all uncountable $I \subset \omega_1$, $\bigcup_{\alpha \in I} h_\alpha$ is not 1-to-1 on any member of x.

Force with $3^{<\mathbb{N}}$ thus adding a partition C_0, C_1, C_2 $(\alpha, \beta) \in R$ (per OCA) if there are $i \in C_0 \cap a_\alpha$, $j \in C_1 \cap a_\beta$ so that $h_\alpha(i) = h_\beta(j) \in C_2$.

$\{\ C_2 \cap h_\alpha(a_\alpha \cap C_0)\ ,\ \ C_2 \cap h_\alpha(a_\alpha \cap C_1) : \alpha \in \omega_1\}$
forms a gap, and if $w \in C_2^*$ is in common closure, there are $x \in C_0^*$ and $y \in C_1^*$ such that $f(x) = f(w) = f(y)$