Moron Maps and subspaces of $\mathbb{N}^*$

what you need to know if you want to work on $\mathbb{N}^*$

and you should!

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Suppose that $f : \mathbb{N}^* \mapsto K$ is precisely 2-to-1 (distinct from $\leq 2$-to-1). What can then be said of $K$ and $f$ (how $\mathbb{N}^*$-like is $K$?)
Connecting Theme

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What are the results, what are the methods needed, and what are the connected questions along the way?
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History to this question (of R. Levy): ?Glazer? and van Douwen’s maximal space
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History to this question (of R. Levy): ?Glazer? and van Douwen’s maximal space

\( E \) is a vD space if there is a 1-to-1 map \( f : \mathbb{N} \rightarrow E \) such that the extension \( f = f^\beta : \beta\mathbb{N} \rightarrow \beta E \) is \( \leq 2\)-to-1; and such a space exists. And \( \beta E \) can be embedded into \( \beta\mathbb{N} \) so that \( f \) is a retract.
[vD] for each $y \in \beta E$, $|f^{-1}(y)| = 1$ iff $y$ is a far point of $E$ (not a limit of any countable (closed) discrete set).
2-to-1 maps

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**Question 1** Does every countable space have a far point? Does every vD space?
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Unfortunately, even if \( E \) had no far points, \( f \upharpoonright \mathbb{N}^* \) is still 1-to-1 at the points of \( f^{-1}(E) \). MA_{ctble} implies all countable spaces have far points.
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we could ask many questions about vD spaces, but the question is about 2-to-1 maps and images of \( \mathbb{N}^* \) (not of \( \beta \mathbb{N} \)). e.g. **Question 2** if \( \mathbb{N}^* \) maps \( \leq 2\text{-to-1} \) onto \( K \subset \mathbb{N}^* \), does the map lift to a \( \leq 2\text{-to-1} \) map on(to) \( \beta \mathbb{N} \)?
Levy’s questions: Let $f : \mathbb{N}^* \leftrightarrow K$ be 2-to-1.

[Levy \implies] countable discrete subsets of $K$ have closures homeomorphic to $\beta \mathbb{N}$. Hence $K$ has weight $\mathfrak{c}$. 
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Item 3 is our starting point for investigation.
Could $f$ be irreducible?

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pull this back to $\mathbb{N}^*$:

Define $J_a = a^* \cap f^{-1}(f[(\mathbb{N} \setminus a)^*])$.

$J_a$ is homeomorphic to $J_{\mathbb{N}\setminus a}$ (via $f^{-1} \circ f$); and both to $f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$. 
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this connects to studied questions about covering \( \mathbb{N}^* \) by nwd sets
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**Question 4** Con(MA + no P-set cover) but PFA or MA?
Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)
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Set $K_1 = K \setminus U_0$ and $X_1 = f^{-1}[K_1]$, hence $f : X_1 \mapsto K_1$ is 2-to-1 (and repeat)
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similarly there is $U_1 \subset K_1$ and $A_1 \oplus_{X_2} B_1$ with $X_2 = f^{-1}[K_2 = (K_1 \setminus U_1)]$
\[ U_0 \approx A_0 \setminus X_1 \]

\[ X_1 = A_1 \cup B_1 \]

\[ \text{if, e.g. } K_2 = \emptyset \]

\[ \text{i.e. } U_1 = K_1 \]

\[ \text{pick clopen set } W \subset N^* \text{ such that } W \cap X_1 = A_1 \]

\[ \text{K is Parovicenko can be shown } \vdash K_n \text{IS empty for some } n \in \omega \]

\[ \text{can all this happen?} \]

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tie-points and propeller points

Say that $x \in \mathbb{N}^*$ is a tie-point if there are closed sets $A, B$ covering $\mathbb{N}^*$ and $\{x\} = A' \cap B'$; denote this as $\mathbb{N}^* = A \oplus_x B$. 

I do not know if it's the same to ask for $x$ such that there is an involution $f$ on $\mathbb{N}^*$ with $\{x\} = \text{fix}(f)$; but I think it is interesting to investigate possible "values" for $\text{fix}(f)$. 

We could further measure $\tau(x) \geq k$ by increasing the number of wings.
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propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$);
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show that $\mathbb{N}^* \approx (\omega \times \{\omega\})^* \approx A \oplus_{\mathbb{N}^*} B$

with $\mathbb{N}^*$ as a propeller set, hence $K_2 \neq \emptyset$ (and iterate)
propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

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my best guess for a $K \not\approx \mathbb{N}^*$ is to have propeller points
$\mathbb{N}^* = A_i \oplus_{x_i} B_i$ so that $A_1 \not\approx \mathbb{N}^*$ and/or $A_1 \oplus_{x_2} B_2 \not\approx \mathbb{N}^*$
some PFA tricks; tie-points; and regular closed sets

Can there be tie-points? and if there are, can $A \approx \mathbb{N}^*$?
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Major open problem: **Question 5** If $f$ embeds $\mathbb{N}^*$ into $\mathbb{N}^*$, is there a lifting from $\beta \mathbb{N}$ to $\beta \mathbb{N}$?
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Major open problem: **Question 5** If $f$ embeds $\mathbb{N}^*$ into $\mathbb{N}^*$, is there a lifting from $\beta\mathbb{N}$ to $\beta\mathbb{N}$?

an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is **ccc over fin** if there is no uncountable almost disjoint family in $\mathcal{I}^+$;
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Major open problem: **Question 5** If $f$ embeds $\mathbb{N}^*$ into $\mathbb{N}^*$, is there a lifting from $\beta \mathbb{N}$ to $\beta \mathbb{N}$?

an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is *ccc over fin* if there is no uncountable almost disjoint family in $\mathcal{I}^+$;

similarly a closed set $K \subset \mathbb{N}^*$ can be said to be ccc over fin if there is no uncountable family of disjoint clopen subsets of $\mathbb{N}^*$ each hitting $K$ (this is more general than requiring that $K$ is contained in a ccc space)
the CH, Cohen + OCA tricks
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Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$
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CH trick $<_{\omega_1 \omega_2} \vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each “proper” extension)
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CH trick plus OCA trick implies no $(\omega_2, \kappa)$-gaps for $\kappa \notin \{1, \omega\}$
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or if each $x_\alpha = h_\alpha : \mathbb{N} \leftrightarrow \mathbb{N}$ is a partial function and $h_\alpha \cup h_\beta$ is not a function for $\alpha \neq \beta$, then there is no common mod finite extension
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so if $\mathcal{H}$ is a coherent family of functions and $\{\text{dom}(h) : h \in \mathcal{H}\}$ is a $P_{\omega_2}$-ideal, then THERE IS a common mod finite extension
forcing a gap from Shelah-Steprans

Start with PFA, use the CH trick to pass to the forcing extension by $\langle \omega_1, \omega_2 \rangle$. This leaves $\mathcal{P}(\mathbb{N})$ unchanged.
forcing a gap from Shelah-Steprans

Start with PFA, use the CH trick to pass to the forcing extension by $<\omega_1\omega_2$. This leaves $\mathcal{P}(\mathbb{N})$ unchanged.

let $Q$ be a ccc poset of cardinality $\omega_1$ and $\{\dot{Y}_\alpha : \alpha \in \omega_1\}$ enumerate all (nice) $Q$-names of subsets of $\mathbb{N}$. 
forcing a gap from Shelah-Steprans

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inductively (or otherwise) choose $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\} \subseteq V \cap \mathcal{P}(\mathbb{N})$, so that, for $\beta < \alpha$, $\models Q \dot{Y}_\beta \cap (c_\alpha \cup d_\alpha) \neq^* c_\alpha$ (if possible: make them $\subset^*$ increasing)
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then in the extension by $Q$, $(\alpha, \beta) \in R$ providing $(c_\alpha \cap d_\beta) \cup (d_\alpha \cap c_\beta) \neq \emptyset$ satisfies that $[X']^2 \cap R$ is not empty for all uncountable $X' \subset X = \omega_1$
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Start with PFA, use the CH trick to pass to the forcing extension by \( <\omega_1 \omega_2 \). This leaves \( \mathcal{P}(\mathbb{N}) \) unchanged.

let \( Q \) be a ccc poset of cardinality \( \omega_1 \) and \( \{ \dot{Y}_\alpha : \alpha \in \omega_1 \} \) enumerate all (nice) \( Q \)-names of subsets of \( \mathbb{N} \).

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then in the extension by \( Q \), \( (\alpha, \beta) \in R \) providing \( (c_\alpha \cap d_\beta) \cup (d_\alpha \cap c_\beta) \neq \emptyset \) satisfies that \( [X']^2 \cap R \) is not empty for all uncountable \( X' \subset X = \omega_1 \) thus this is a freezable gap: no \( Y \) such that \( Y \cap (c_\alpha \cup d_\alpha) =^* c_\alpha \) for all \( \alpha \).
ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin
ccc over fin boundaries; per 2-points and embeddings

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Remark: CH implies every closed nowhere dense set is a boundary of a regular closed set
ccc over fin boundaries; per 2-points and embeddings

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Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin.

Let \( A \subseteq \mathbb{N}^* \) be regular closed. So \( I \cup J \) is dense, where \( a \in I \) if \( a^* \subset A \) and \( b \in J \) if \( b^* \cap A = \emptyset \).
Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

Lemma: $\partial A$ is ccc over fin implies $\mathcal{I}$ and $\mathcal{J}$ are P-ideals
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**Lemma:** $\partial A$ is ccc over fin implies $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

let $\{a_n : n \in \omega\} \subset \mathcal{I}$ be pairwise disjoint;
Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

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let $\{a_n : n \in \omega\} \subset \mathcal{I}$ be pairwise disjoint;

for each $g \in \mathbb{N}^\omega$, let $L_g = \bigcup_n a_n \cap g(n)$. 
Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $I \cup J$ is dense, where $a \in I$ if $a^* \subset A$ and $b \in J$ if $b^* \cap A = \emptyset$

Lemma: $\partial A$ is ccc over fin implies $I$ and $J$ are P-ideals

let $\{a_n : n \in \omega\} \subset I$ be pairwise disjoint;

for each $g \in \mathbb{N}^\omega$, let $L_g = \bigcup_n a_n \cap g(n)$.

since $\partial A$ is ccc over fin there is an $f \in \mathbb{N}^\omega$ so that $\partial A \cap (L_g \setminus L_f)^*$ is empty for all $g \in \mathbb{N}^\omega$
Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^\ast$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

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so pick, for each $g$, $h_g : L_g \setminus L_f \mapsto 2$ so that $h_g^{-1}(0) \in \mathcal{I}$ and $h_g^{-1}(1) \in \mathcal{J}$. 
\( \mathcal{I} \) and \( \mathcal{J} \) are P-ideals
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the \( L_g \)'s range over a \( P_{\omega_2} \)-ideal so
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let \( h : \mathbb{N} \setminus L_f \mapsto 2 \) mod finite extend \( h_g \) for all \( g \in \mathbb{N}^\omega \)

ccc argument we’ll see later
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with \(b = h^{-1}(1)\) and \(J \subset \omega\) such that \(b \cap a_n\) is infinite for each \(n\),
we have that \(\partial A \cap (b \cap \bigcup_{n \in J} a_n)^*\) is not empty;
since ccc over fin implies such a \(J\) must be finite, we finish that each of \(\mathcal{I}\) and \(\mathcal{J}\) are P-ideals
now is time for CH * Cohen * OCA trick

we continue with proof that $\partial A$ is not ccc over \text{fin}
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we continue with proof that $\partial A$ is not ccc over fin

let $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$ be disjoint pairs from $\mathcal{I} \times \mathcal{J}$ chosen so as to be cofinal.
now is time for $\text{CH} \ast \text{Cohen} \ast \text{OCA}$ trick

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the technique (again, later) produces a proper poset collection of names 1-to-1 $\check{\rho} : 2^{<\omega} \mapsto \mathbb{N}$ and

$\{\alpha(f, \xi) : \xi \in \omega_1, f \in V \cap 2^\omega\} \subset \omega_1$
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so that $\exists n = n(f, \xi, \eta), k = k(f, \xi, \eta)$ satisfying

$$\dot{\rho}(f \upharpoonright k) = n \in (a_\alpha(f, \xi) \cap b_\alpha(f, \eta)) \cup (a_\alpha(f, \eta) \cap b_\alpha(f, \xi))$$
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we continue with proof that \( \partial A \) is not ccc over \( \text{fin} \)

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so that \( \exists n = n(f, \xi, \eta), k = k(f, \xi, \eta) \) satisfying
\[
\dot{\rho}(f \upharpoonright k) = n \in (a_\alpha(f, \xi) \cap b_\alpha(f, \eta)) \cup (a_\alpha(f, \eta) \cap b_\alpha(f, \xi))
\]

set \( C_f = \{\rho(f \upharpoonright k) : k \in \omega\} \subset \mathbb{N} \), and \( \Gamma_f = \{\alpha(f, \xi) : \xi \in \omega_1\} \)
continued
fix any “generic” filter meeting $\omega_1$-many dense subsets of this iteration of proper posets, so as to have some uncountable $\mathcal{F} \subset 2^{<\omega}$ and $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$, and values for $\alpha(f, \xi), n(f, \xi, \eta), k(f, \xi, \eta)$ for all $f \in \mathcal{F}$ and $\xi \in \omega_1$. 
fix any “generic” filter meeting $\omega_1$-many dense subsets of this iteration of proper posets, so as to have some uncountable $\mathcal{F} \subset 2^{<\omega}$ and $\{ (a_\alpha, b_\alpha) : \alpha \in \omega_1 \} \subset \mathcal{I} \times \mathcal{J}$, and values for $\alpha(f, \xi), n(f, \xi, \eta), k(f, \xi, \eta)$ for all $f \in \mathcal{F}$ and $\xi \in \omega_1$.

so that for $f \in \mathcal{F}$, $\xi \neq \eta \in \omega_1$,

$$\rho(f \upharpoonright k) = n \in (a_{\alpha(f, \xi)} \cap b_{\alpha(f, \eta)}) \cup (a_{\alpha(f, \eta)} \cap b_{\alpha(f, \xi)})$$
fix any “generic” filter meeting $\omega_1$-many dense subsets of this iteration of proper posets, so as to have some uncountable $F \subset 2^{<\omega}$ and $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\} \subset I \times J$, and values for $\alpha(f, \xi), n(f, \xi, \eta), k(f, \xi, \eta)$ for all $f \in F$ and $\xi \in \omega_1$.

so that for $f \in F$, $\xi \neq \eta \in \omega_1$,

$$\rho(f \upharpoonright k) = n \in (a_{\alpha(f, \xi)} \cap b_{\alpha(f, \eta)}) \cup (a_{\alpha(f, \eta)} \cap b_{\alpha(f, \xi)})$$

we obtain that $C_f^* \cap \partial A$ is non-empty for all $f \in F$ because

$$\partial A \supset \bigcup_{\alpha \in \Gamma_f} (a_\alpha \cap C_f)^* \cap \bigcup_{\alpha \in \Gamma_f} (b_\alpha \cap C_f)^* \neq \emptyset$$
okay, we freeze a gap
okay, we freeze a gap

we have the gap \( \{a_\alpha, b_\alpha : \alpha \in \omega_1\} \); mod finite increasing and \( a_\alpha \cap b_\alpha \) empty. (enough that \( a_\alpha \)'s increase)
okay, we freeze a gap

we have the gap \{a_\alpha, b_\alpha : \alpha \in \omega_1\}; mod finite increasing and \(a_\alpha \cap b_\alpha\) empty. (enough that \(a_\alpha\)'s increase)

a pair \((\rho, H) \in Q\) if there is an \(n\) with \(\rho : 2^{<n} \leftrightarrow 1 \rightarrow \omega \rightarrow \mathbb{N}\), and \(H \in [\omega_1]^{<\omega}\) is such that

for each \(\alpha \neq \beta \in H\), and each \(t \in 2^n\), there is a \(k < n\) with \(\rho(t \upharpoonright k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)\)
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a pair \((\rho, H) \in Q\) if there is an \(n\) with \(\rho : 2^{<n} \overset{1 \leftrightarrow 1}{\rightarrow} \mathbb{N}\), and \(H \in [\omega_1]^{<\omega}\) is such that

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assume \(\{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q\); and that \(H_\xi \cap H_\eta = H\) for all \(\xi \neq \eta \in \omega_1\); and pairwise "isomorphic"

set \(A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha\) and \(B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha\)
okay, we freeze a gap

we have the gap \( \{a_\alpha, b_\alpha : \alpha \in \omega_1\} \); mod finite increasing and \( a_\alpha \cap b_\alpha \) empty. (enough that \( a_\alpha \)'s increase)

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\( H \in [\omega_1]^{<\omega} \) is such that

for each \( \alpha \neq \beta \in H \), and each \( t \in 2^n \), there is a \( k < n \) with
\( \rho(t \upharpoonright k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \)

assume \( \{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q \); and that \( H_\xi \cap H_\eta = H \) for all \( \xi \neq \eta \in \omega_1 \); and pairwise “isomorphic”

set \( A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha \) and \( B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha \) Let \( l_0 = J_0 = \omega_1 \), \( S_0 = \{i : \exists \omega^1 \xi \in l_0 \ i \in A_\xi\} \) and \( T_0 = \{i : \exists \omega^1 \eta \in J_0 \ i \in B_\xi\} \)
okay, we freeze a gap

we have the gap \( \{ a_\alpha, b_\alpha : \alpha \in \omega_1 \} \); mod finite increasing and \( a_\alpha \cap b_\alpha \) empty. (enough that \( a_\alpha \)'s increase)

a pair \((\rho, H) \in Q \) if there is an \( n \) with \( \rho : 2^{<n} \to \omega \), and \( H \in [\omega_1]^{<\omega} \) is such that

for each \( \alpha \neq \beta \in H \), and each \( t \in 2^n \), there is a \( k < n \) with 
\( \rho(t \upharpoonright k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \)

assume \( \{ (\rho, H_\xi) : \xi \in \omega_1 \} \subset Q; \) and that \( H_\xi \cap H_\eta = H \) for all \( \xi \neq \eta \in \omega_1; \) and pairwise “isomorphic”

set \( A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha \) and \( B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha \). Let \( l_0 = J_0 = \omega_1 \), \( S_0 = \{ i : \exists \omega_1 \xi \in l_0 \ i \in A_\xi \} \) and \( T_0 = \{ i : \exists \omega_1 \eta \in J_0 \ i \in B_\xi \} \)

there is \( i_0 \in S_0 \cap T_0; \) set \( l_1 = \{ \xi \in l_0 : i_0 \in A_\xi \} \);
\( J_1 = \{ \eta \in J_0 : i_0 \in B_\eta \} \)
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a pair \((\rho, H) \in Q\) if there is an \( n \) with \( \rho : 2^{<n} \rightarrow^{1} \mathbb{N} \), and \( H \in [\omega_1]^{<\omega} \) is such that

for each \( \alpha \neq \beta \in H \), and each \( t \in 2^n \), there is a \( k < n \) with \( \rho(t \upharpoonright k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \)

assume \( \{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q \); and that \( H_\xi \cap H_\eta = H \) for all \( \xi \neq \eta \in \omega_1 \); and pairwise “isomorphic”

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there is \( i_0 \in S_0 \cap T_0 \); set \( l_1 = \{\xi \in l_0 : i_0 \in A_\xi\} \);
\( J_1 = \{\eta \in J_0 : i_0 \in B_\eta\} \) repeat \( 2^n \) times getting \( \{i_t\}_{t \in 2^n} \)