Moron Maps and subspaces of N*

what you need to know if you want to work on N*

and you should!

Alan Dow

Department of Mathematics
University of North Carolina Charlotte

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Connecting Theme

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E is a vD space if there is a 1-to-1 map $f : \mathbb{N} \mapsto E$ such that the extension $f = f^\beta : \beta\mathbb{N} \mapsto \beta E$ is $\leq$2-to-1; and such a space exists. And βE can be embedded into $\beta\mathbb{N}$ so that f is a retract.

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Item 3 is our starting point for investigation.

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this connects to studied questions about covering $\mathbb{N}^*$ by nwd sets

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Question 4 Con(MA + no P-set cover) but PFA or MA$\vdash$?

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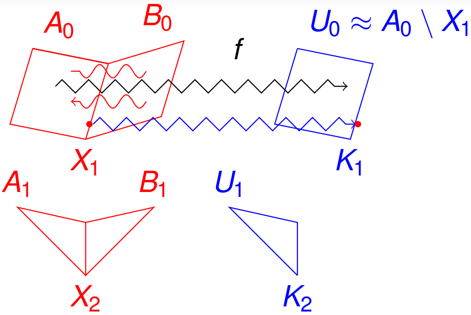
similarly there is $U_1 \subset K_1$ and $A_1 \oplus_{X_2} B_1$ with $X_2 = f^{-1}[K_2 = (K_1 \setminus U_1)]$



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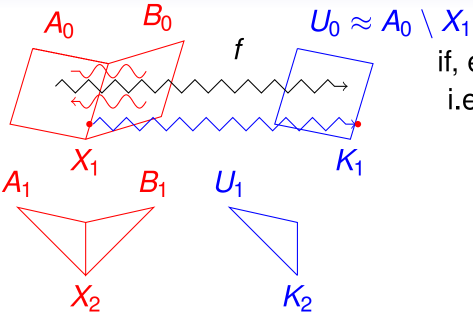

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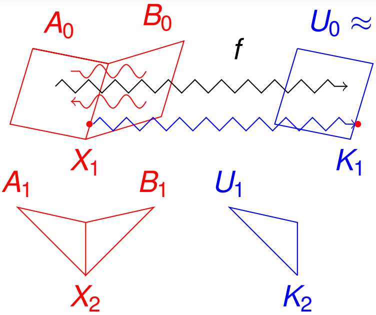

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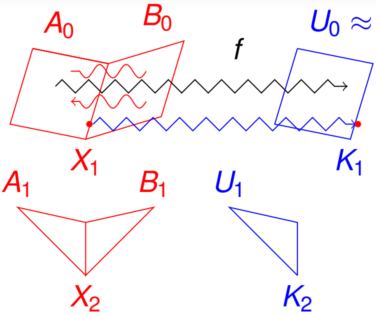

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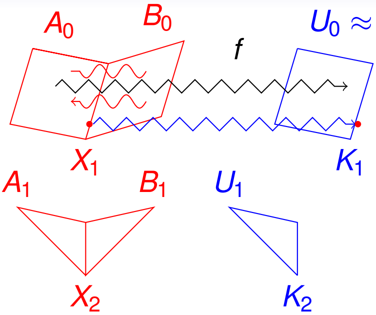

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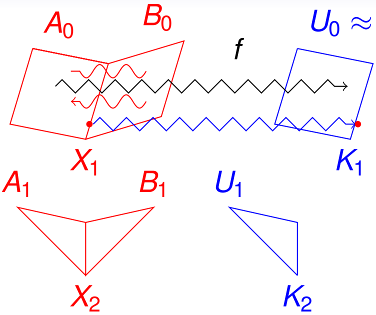

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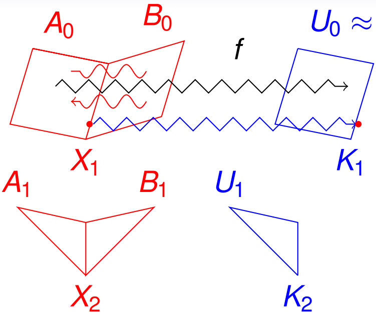

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THUS CH implies $K \approx \mathbb{N}^*$

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I do not know if it's the same to ask for x such that there is an involution f on $\mathbb{N}^*$ with $\{x\} = \mathrm{fix}(f)$; but I think it is interesting to investigate possible "values" for $\mathrm{fix}(f)$

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my best guess for a $K \not\approx \mathbb{N}^*$ is to have propeller points $\mathbb{N}^* = A_i \oplus_{x_i} B_i$ so that $A_1 \not\approx \mathbb{N}^*$ and/or $A_1 \oplus^{x_1}_{x_2} B_2 \not\approx \mathbb{N}^*$

some PFA tricks; tie-points; and regular closed sets

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similarly a closed set $K \subset \mathbb{N}^*$ can be said to be ccc over fin if there is no uncountable family of disjoint clopen subsets of $\mathbb{N}^*$ each hitting K (this is more general than requiring that K is contained in a ccc space)

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so if $\mathcal{H}$ is a coherent family of functions and $\{\mathrm{dom}(h) : h \in \mathcal{H}\}$ is a P_{ω_2}-ideal, then THERE IS a common mod finite extension

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Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

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Remark: CH implies every closed nowhere dense set is a boundary of a regular closed set

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so pick, for each g, $h_g : L_g \setminus L_f \mapsto 2$ so that $h_g^{-1}(0) \in \mathcal{I}$ and $h_g^{-1}(1) \in \mathcal{J}$.

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with $b = h^{-1}(1)$ and $J \subset \omega$ such that $b \cap a_n$ is infinite for each n, we have that $\partial A \cap \left(b \cap \bigcup_{n \in J} a_n\right)^*$ is not empty;
since ccc over fin implies such a J must be finite, we finish that each of $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

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the technique (again, later) produces a proper poset collection of names 1-to-1 $\dot{\rho} : 2^{<\omega} \mapsto \mathbb{N}$ and
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so that $\exists n = n(f,\xi,\eta), k = k(f,\xi,\eta)$ satisfying

$$\dot\rho(f \restriction k) = n \in (a_{\alpha(f,\xi)} \cap b_{\alpha(f,\eta)}) \cup (a_{\alpha(f,\eta)} \cap b_{\alpha(f,\xi)})$$

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let $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$ be disjoint pairs from $\mathcal{I} \times \mathcal{J}$ chosen so as to be cofinal.

the technique (again, later) produces a proper poset collection of names 1-to-1 $\dot\rho : 2^{<\omega} \mapsto \mathbb{N}$ and $\{\alpha(f,\xi) : \xi \in \omega_1, f \in V \cap 2^\omega\} \subset \omega_1$

so that $\exists n = n(f,\xi,\eta), k = k(f,\xi,\eta)$ satisfying

$$\dot\rho(f \restriction k) = n \in (a_{\alpha(f,\xi)} \cap b_{\alpha(f,\eta)}) \cup (a_{\alpha(f,\eta)} \cap b_{\alpha(f,\xi)})$$

set $C_f = \{\rho(f \restriction k) : k \in \omega\} \subset \mathbb{N}$, and $\Gamma_f = \{\alpha(f,\xi) : \xi \in \omega_1\}$

continued

continued

fix any “generic” filter meeting ω_1-many dense subsets of this iteration of proper posets, so as to have some uncountable $\mathcal{F} \subset 2^{<\omega}$ and $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$, and values for $\alpha(f,\xi), n(f,\xi,\eta), k(f,\xi,\eta)$ for all $f \in \mathcal{F}$ and $\xi \in \omega_1$.

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so that for $f \in \mathcal{F}$, $\xi \neq \eta \in \omega_1$,

$$\rho(f \restriction k) = n \in (a_{\alpha(f,\xi)} \cap b_{\alpha(f,\eta)}) \cup (a_{\alpha(f,\eta)} \cap b_{\alpha(f,\xi)})$$

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fix any “generic” filter meeting ω_1-many dense subsets of this iteration of proper posets, so as to have some uncountable $\mathcal{F} \subset 2^{<\omega}$ and $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$, and values for $\alpha(f,\xi), n(f,\xi,\eta), k(f,\xi,\eta)$ for all $f \in \mathcal{F}$ and $\xi \in \omega_1$.

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we obtain that $C_f^* \cap \partial A$ is non-empty for all $f \in \mathcal{F}$ because

$$\partial A \supset \overline{\bigcup_{\alpha \in \Gamma_f} (a_\alpha \cap C_f)^*} \cap \overline{\bigcup_{\alpha \in \Gamma_f} (b_\alpha \cap C_f)^*} \neq \emptyset$$

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we have the gap $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$; mod finite increasing and $a_\alpha \cap b_\alpha$ empty. (enough that a_α's increase)

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a pair $(\rho, H) \in Q$ if there is an n with $\rho : 2^{<n} \overset{1-1}{\mapsto} \mathbb{N}$, and $H \in [\omega_1]^{<\omega}$ is such that

for each $\alpha \neq \beta \in H$, and each $t \in 2^n$, there is a $k < n$ with $\rho(t \restriction k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$

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assume $\{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q$; and that $H_\xi \cap H_\eta = H$ for all $\xi \neq \eta \in \omega_1$; and pairwise "isomorphic"

set $A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha$ and $B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha$

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set $A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha$ and $B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha$ Let $I_0 = J_0 = \omega_1$, $S_0 = \{i : \exists^{\omega_1} \xi \in I_0 \;\; i \in A_\xi\}$ and $T_0 = \{i : \exists^{\omega_1} \eta \in J_0 \;\; i \in B_\xi\}$

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there is $i_0 \in S_0 \cap T_0$; set $I_1 = \{\xi \in I_0 : i_0 \in A_\xi\}$; $J_1 = \{\eta \in J_0 : i_0 \in B_\eta\}$

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there is $i_0 \in S_0 \cap T_0$; set $I_1 = \{\xi \in I_0 : i_0 \in A_\xi\}$; $J_1 = \{\eta \in J_0 : i_0 \in B_\eta\}$ repeat 2^n times getting $\{i_t\}_{t \in 2^n}$