Extending the Classical Results on Club Guessing

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Soon after Jensen defined $\diamondsuit$, Ostaszewski formulated the weaker $\clubsuit$ principle. Going by the definition of $\diamondsuit$ that we have chosen to use, $\clubsuit$ can be thought of as simply $\diamondsuit$ with subsethood replacing equality.

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In some ways, although it is less widely used, $\clubsuit$ is more interesting than $\diamondsuit$; it has the advantage of removing the cardinal arithmetic assumptions that are latent in $\diamondsuit$. For example, $\clubsuit(\omega_1)$ is consistent with a large continuum whereas $\diamondsuit(\omega_1) \to \mathrm{CH}$.

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Proof.

For any $x \subseteq \omega$ there will be some $X \in [\omega_1]^{\omega_1}$ such that $X \cap \omega = x = B_\delta \cap \omega$ for some $\omega < \delta < \omega_1$. Hence $\langle B_\delta \cap \omega : \delta < \omega_1 \rangle$ enumerates the continuum.

□

What is meant by a "guessing sequence"...

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Another reason guessing sequences are worth investigating is that some non-trivial ones exist in ZFC! A seminal result in combinatorial set theory is Shelah's proof of *club guessing* for regular cardinals greater than $\aleph_1$.

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We say that $\clubsuit_{\mathcal{X}}(S)$ *holds, for* $\mathcal{X} \subseteq [\lambda]^{\lambda}$ *and* $S \subseteq \lambda$ *stationary, if there exists a sequence* $\langle A_\delta : \delta \in S \rangle$ *such that* $A_\delta \subseteq \delta$ *is unbounded and for all* $X \in \mathcal{X}$ *the set* $\{\delta \in S : A_\delta \subseteq X\}$ *is stationary.*

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Definition

We write S^λ_κ if κ and λ are regular for the (stationary) set $\{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$. This is sometimes written E^λ_κ, e.g. in Jech.

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Can anything non-trivial be said about $\clubsuit_{\mathrm{STAT}}$, where STAT denotes the set of stationary subsets of λ?
For example, does $\clubsuit_{\mathrm{STAT}} \to \clubsuit$?
Does $\clubsuit_{\mathrm{STAT}}(\omega_1) + \mathrm{CH} \to \diamondsuit(\omega_1)$?

A Possible Partial Answer...

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Theorem (Primavesi, 2009)

Let λ be uncountable. If $\mathcal{F}$ is a uniform λ^+-closed filter on λ^+, then $\clubsuit_{\mathcal{F}}(S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)})$ holds in ZFC.

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The proof is based on Shelah's proof, but we cannot rely on starting with local clubs. We have to do a bit more work at the start. We leave this as an exercise.