Handbook of Set Theory

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Contents

I. Cardinal Arithmetic

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1. Introduction

Cardinal arithmetic is the study of rules and properties of arithmetic operations mainly on infinite cardinal numbers. Since sums and products are trivial in the sense that

$$m + n = m \cdot n = \max\{m, n\}$$

holds for infinite cardinals, cardinal arithmetic refers mainly to exponentiation m^n. If M is a set of cardinality m and $[M]^n$ is the collection of all subsets of M of cardinality n, then m^n is equal to the cardinality of $[M]^n$. Thus exponentiation is intimately connected with the power-set operation and hence lies at the heart of set theory. Classical and basic properties of cardinal arithmetic can be found, for example, in the Levy [12] and Jech [8] textbooks (the latter contains more advanced material). The aim of this introduction is to mention some elementary results and to put our chapter in its context—not to give a historical introduction to the subject of cardinal arithmetic, for which the reader is referred to these textbooks, to [7], and to [9] for a more general perspective.

A theorem of Zermelo generalizing a result of J. König says that if $\langle \kappa_i \mid i \in I \rangle$ and $\langle \lambda_i \mid i \in I \rangle$ are sequences of cardinals such that $\kappa_i < \lambda_i$ holds for every $i \in I$, then

$$\Sigma_{i \in I} \kappa_i < \Pi_{i \in I} \lambda_i.$$

A theorem of Bukovský and of Hechler says that if μ is a singular cardinal and the values 2^γ for cardinals $\gamma < \mu$ stabilize, then $2^\mu = 2^{\gamma_0}$, where $\gamma_0 < \mu$ is such that $2^{\gamma_0} = 2^\gamma$ for all $\gamma_0 \leq \gamma < \mu$.

Building on earlier results (of Hausdorff, Tarski, Bernstein and others) Bukovský (1965) and Jech show how cardinal exponentiation can be computed from the gimel function (which takes κ to $\kappa^{\mathrm{cf}(\kappa)}$). Applications of Solovay and Easton of the forcing method of Cohen (1963) show that for regular cardinals κ there is no restriction on 2^κ except that which follows from the Zermelo–König theorem, namely that $\mathrm{cf}(2^\kappa) > \kappa$ (see [8] Chapter 3 for details). Thus the question about the possible values of $\kappa^{\mathrm{cf}(\kappa)}$ is most interesting from our point of view when κ is a singular cardinal. It was evident that it is much harder to apply the forcing method to singular cardinals. Involving large cardinals, work of Prikry and of Silver showed that it is possible for a strong limit singular cardinal μ to satisfy $2^\mu > \mu^+$ in some generic extension. Using large cardinals Magidor proved the consistency of $\aleph_\omega$ being the first cardinal κ for which $2^\kappa > \kappa^+$ holds. For a long time it was believed that large cardinal and more complex applications of the forcing method should yield greater flexibility for values of the power-set of singular cardinals. A first indication that there are possible limitations was the theorem of Silver (1974): If κ is a singular cardinal with uncountable cofinality and if $2^\delta = \delta^+$ for all cardinals $\delta < \kappa$, then $2^\kappa = \kappa^+$. This result paved the way for further investigations by Galvin and Hajnal (1975). A representative result of their work is the following: If $\aleph_\delta$ is a strong limit singular cardinal with uncountable cofinality, then

$$2^{\aleph_\delta} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}.$$

For example, if $\aleph_{\omega_1}$ is a strong limit cardinal, then

$$2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}.$$

The method of proof of these results relied in an essential way on the assumption that $\mathrm{cf}(\delta) > \aleph_0$. Shelah (1978) was able to prove similar results for singular cardinals with countable cofinality. For example, if $\aleph_\omega$ is a strong limit cardinal, then

$$2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}. \tag{I.1}$$

In a series of papers, culminating in his book [14], Shelah developed a powerful theory with many applications, the pcf theory, which changed our view of cardinal arithmetic. A remarkable result of this theory is the following. If $\aleph_\omega$ is a strong limit cardinal then

$$2^{\aleph_\omega} < \aleph_{\omega_4}. \tag{I.2}$$

If $2^{\aleph_0} \leq \aleph_2$, then (I.1) is a better bound than (I.2), but in general, since $(2^{\aleph_0})^+$ can be arbitrarily high, ω_4 seems to be a firmer bound.

The major definition in pcf theory is the set $\mathrm{pcf}(A)$ of possible cofinalities defined for every set A of regular cardinals as the collection of all cofinalities of ultraproducts $\Pi A/D$ with ultrafilters D over A. This basic and rather simple definition appears in many places and is the basis of a very fruitful investigation. It is a basic definition also in the sense that while the power-set can be easily changed by forcing, it is very hard to change $\mathrm{pcf}(A)$.

Our aim in this chapter is to give a self-contained development of pcf theory and to present some of its important applications to cardinal arithmetic. Unless stated otherwise, all theorems and results in this chapter are due to Shelah.

The fullest development of the pcf theory is in Shelah's book [14], and the interested reader can access newer articles (and the survey paper "Analytical Guide") in the archive maintained at Rutgers University.

In addition to this material, we have profited from expository papers (Burke–Magidor [2], Jech [7], and unpublished notes by Hajnal), and in particular a recently published book [6] which is very detailed, complete and carefully written.

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2. Elementary definitions

An *ideal* over a set A is a collection $I \subseteq \mathcal{P}(A)$ such that (1) I is closed under subsets, that is, $X \in I$ and $Y \subseteq X$ implies $Y \in I$, and (2) I is closed under finite unions, that is, $X_1, X_2 \in I$ imply $X_1 \cup X_2 \in I$ (and thus the union of any finite sequence of members of I is in I). If $A \notin I$, then I is said to be *proper*. We do not require that ideals be proper (see the definition of $J_{<\lambda}$ in section 3.1).

The dual notion, that of a filter, is also used in this chapter. A collection $F \subseteq \mathcal{P}(A)$ is a *filter* over A if (1) F is closed under supersets, that is, $X \in F$ and $X \subseteq Y \subseteq A$ imply $Y \in F$, and (2) F is closed under finite intersections. However, usually a filter is proper, that is $\emptyset \notin F$.

If I is an ideal over A, then $I^* = \{X \subseteq A \mid A \setminus X \in I\}$ is its *dual* filter. Sets belonging to an ideal are intuitively "small" or "null", whereas those of a filter are "big" or "of measure one". If I is an ideal over A, then subsets of A not in I are called "positive", and the collection of positive sets is denoted I^+.

$$I^+ = \{X \subseteq A \mid X \notin I\}.$$

We shall deal in this section with functions from a fixed, infinite set A into the ordinals. The class of these ordinal functions is denoted On^A. If $f, g \in \mathrm{On}^A$, then $f \leq g$ means that $f(a) \leq g(a)$ for all $a \in A$, and similarly $f < g$ means that $f(a) < g(a)$ for all $a \in A$ (this is called the everywhere dominance ordering).

If $F \subset \mathrm{On}^A$ is a set, then the supremum function $h = \sup F$ is defined on A by

$$h(a) = \sup\{f(a) \mid f \in F\}.$$

If $f, g \in \mathrm{On}^A$, then we define

$$<(f,g) = \{a \in A \mid f(a) < g(a)\},$$

and similarly

$$\leq(f,g) = \{a \in A \mid f(a) \leq g(a)\}.$$

and

$$=(f,g) \;=\; \{a \in A \mid f(a) = g(a)\}.$$

If I is an ideal over A, then we define a relation $\leq_I$ over On^A by

$$f \leq_I g \quad \text{iff} \quad \{a \in A \mid g(a) < f(a)\} \in I.$$

In general, for any relation R on the ordinals, we define R_I over On^A by

$$f \; R_I \; g \text{ iff } \{a \in A \mid \neg(f(a) \; R \; g(a))\} \in I.$$

That is, the set of exceptions to the relation is null. In this way $<_I$ and $=_I$ are defined over On^A. We remark that $\leq_I$ is weaker than "$<_I$ or $=_I$". Note that $\leq_I$ is a quasi-ordering, and that $<_I$ is irreflexive (if I is a proper ideal) and transitive.

The notations $X \subseteq_I Y$ and $X =_I Y$ are also used for subsets $X, Y \subseteq A$, in the obvious meaning. For example, $X \subseteq_I Y$ iff $X \setminus Y \in I$.

For a filter F over A, the dual definitions $f <_F g$, $f \leq_F g$ etc. will be used as well. For example, $f <_F g$ means that $\{a \in A \mid f(a) < g(a)\} \in F$. If F is the dual of an ideal I, then $<_F$ and $<_I$ are the same relation of course.

Products of sets

Suppose that A is an index set and $S = \langle S_a \mid a \in A \rangle$ is a sequence of non-empty sets of ordinals. Then the product, denoted ΠS or $\Pi_{a \in A} \, S_a$, is defined as

$$\Pi S = \{f \mid f \in \mathrm{On}^A \text{ and } \forall a \in A \; f(a) \in S_a\}.$$

In particular, if $h : A \to \mathrm{On}$ is any ordinal function defined on A, then Πh (or $\Pi_{a \in A} \, h(a)$) denotes the set of all ordinal functions f defined on A such that $f(a) \in h(a)$ for all $a \in A$.

If A is a set of cardinals, then ΠA (or $\Pi_{a \in A} \; a$) denotes the set of all ordinal functions f defined on A such that $f(a) \in a$ for all $a \in A$. That is, ΠA is Πh where $h(a) = a$ is the identity function on A.

For an ideal I over A, the relations $<_I$, $\leq_I$, and $=_I$ are defined on Πh, and the reduced product $\Pi h / I$ consisting of all $=_I$ equivalence classes is obtained. If $g \in \mathrm{On}^A$, then we may write (somewhat informally) $g \in \Pi h / I$ rather than $[g] \in \Pi h / I$, that is $\Pi h / I$ is considered as a class of functions rather than equivalence classes.

For a filter F over A, the reduced product $\Pi h/F$ is defined in a similar way.

A sequence of functions $f = \langle f_\xi \mid \xi < \lambda \rangle$ in ΠA is said to be $<_I$-increasing if $\xi_1 < \xi_2$ implies that $f_{\xi_1} <_I f_{\xi_2}$. For typographical reasons we also say that f is I-increasing, or "increasing modulo I" instead of $<_I$-increasing. A sequence is a function, and if f_ξ denotes a value of that function then the sequence itself is denoted f, not $\bar{f}$ or F.

Partial orderings

We say that $(P, \leq_P)$ is a quasi-ordering iff $\leq_P$ is a reflexive and transitive relation on P. A *strict* partial ordering is a transitive and irreflexive relation $<_P$ on P. In this chapter we consider both the quasi-ordering $\leq_I$ and the strict partial ordering $<_I$, defined on ordinal valued functions. A typical example is $P = \Pi h$ with the orderings $<_I$ and $\leq_I$ where $h \in \mathrm{On}^A$ is such that $h(a) > 0$ is a limit ordinal for every $a \in A$. Thus every function in P is $<_I$ bounded by another function there (for every $f \in \Pi h$, $f <_I f + 1$, where $f + 1$ is the function taking a to $f(a) + 1$). So our setting is a structure $(P, <_P, \leq_P)$ where $<_P$ is a strict partial ordering, and $\leq_P$ is a quasi-ordering. The following properties of $(P, <_P, \leq_P)$ are obvious for our typical example:

P1 $a <_P b$ or $a = b$ implies $a \leq_P b$, but this implication is not necessarily reversible.

P2 If $a <_P b \leq_P c$ or $a \leq_P b <_P c$, then $a <_P c$.

P3 There is no $<_P$ maximal member: for every $p \in P$ there exists some $p' \in P$ with $p <_P p'$.

The following definitions apply whenever P is a set or a class, $<_P$ is a strict partial ordering, and $\leq_P$ a quasi-ordering on P.

A collection $B \subseteq P$ is said to be *cofinal* in P iff for all $x \in P$ there is some $y \in B$ with $x \leq_P y$. B is $<_P$-cofinal if $\forall x \in P \exists y \in B (x <_P y)$. If B is cofinal and $p \in P$, then we can first find $p' \in P$ such that $p <_P p'$ (by property P3 above) and then find $y \in B$ such that $p' \leq_P y$. Then $p <_P y$. Thus we can replace $\leq_P$ with $<_P$ in the definition of "cofinal". The cofinality, $\mathrm{cf}(P, \leq_P)$, of the partial ordering set is the smallest cardinality of a cofinal subset. (Again $\mathrm{cf}(P, <_P)$ is similarly defined and these two cardinals are equal if properties P1–P3 above hold.) This cardinal needs not be regular, if the ordering is not total (linear). We say that $(P, <_P)$ has "true" cofinality if it has a totally ordered subset $B \subseteq P$ that is cofinal. In this case the cofinality of B itself, and hence of P, is a regular cardinal. Observe that if $(P, <)$ has a linear cofinal subset whose order-type is a regular cardinal λ, then λ is the cofinality of P (because no cofinal subset of

P is of smaller cardinality, even if non-linear subsets are considered). When $(P, <_P)$ has true cofinality, we write

$$\mathrm{tcf}\,(P, <_P) = \lambda$$

to express both the fact that a totally ordered cofinal set exists, and that λ is the minimal cardinality of such a cofinal set.

In cases (when P3 above is not assumed) that $(P, \leq_P)$ has a greatest element, then the cofinality of P is defined to be 1 and its true cofinality is also 1, but since we assume that there are no $<_P$ maximal elements the cofinality and true cofinality (when it exists) are always infinite cardinals.

The following observation was made by Pouzet. For any infinite cardinal λ, $\mathrm{tcf}\,(P, <_P) = \lambda$ if and only if the following conditions hold:

1. $(P, <_P)$ has a cofinal set of size λ.

2. $(P, <_P)$ is λ-directed: any set $X \subseteq P$ of size $< \lambda$ has an upper bound in $(P, <_P)$.

It follows that if $\mathrm{tcf}(P, <_P) = \lambda$ and $G \subseteq P$ is any cofinal subset, then $\mathrm{tcf}\, G = \lambda$ as well.

A sequence $\langle p_\xi \mid \xi < \lambda \rangle$ of members of P is defined to be *persistently cofinal* iff

$$\forall h \in P\ \exists \xi_0 < \lambda\ \forall \xi\ (\xi_0 \leq \xi < \lambda \implies h <_P p_\xi). \tag{I.3}$$

Clearly every $<_P$ increasing and cofinal sequence is persistently cofinal. If $\langle p_\xi \mid \xi < \lambda \rangle$ is persistently cofinal and $p_\xi \leq_P p'_\xi$ for every $\xi < \lambda$, then $\langle p'_\xi \mid \xi < \lambda \rangle$ is persistently cofinal as well.

If $(P, \leq_P)$ is a quasi-ordering and $X \subseteq P$, then an *upper bound* of X is some $a \in P$ such that $x \leq_P a$ for all $x \in X$. If a is an upper bound of X and $a \leq_P a'$ for every upper bound $a' \in P$ of X, then we say that a is a *least upper bound* of X. We say that an upper bound a of X is a *minimal upper bound* if there is no upper bound a' of X such that $a' \leq_P a \wedge \neg(a \leq_P a')$.

Suppose that $(P, <_P, \leq_P)$ is as above and $X \subseteq P$ is such that for every $x \in X$ there is $x' \in X$ with $x <_P x'$ (for example X is an increasing sequence in $<_P$). Then $a \in P$ is an *exact upper bound* of X iff

1. a is a least upper bound of X, and

2. X is cofinal in $\{p \in P \mid p <_P a\}$. Namely $p <_P a$ implies $\exists x \in X (p \leq_P x)$.

Exercises are natural places to stop and think, but it is not an absolute requirement to solve them on first encounter. In fact, they often become easy with later material.

2.1 Exercise. Let $\lambda > |A|$ be a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda \rangle$ an increasing sequence of functions in On^A in the $<$ ordering (of everywhere dominance). Then f has an exact upper bound h and $\mathrm{cf}(h(a)) = \lambda$ for every $a \in A$. In fact $\sup f$ is the required upper bound.

We repeat the definitions given above, for $(\mathrm{On}^A, <_I, \leq_I)$ where I is a proper ideal over A. So, if $F \subset \mathrm{On}^A$ then

$$h \in \mathrm{On}^A \text{ is an upper bound of } F \text{ iff } f \leq_I h \text{ for every } f \in F.$$

A function h is a least upper bound of F if it is an upper bound and $h \leq_I h'$ for every upper bound $h' \in \mathrm{On}^A$ of F. Here, the notions of least upper bound and minimal upper bound coincide.

If $h \in \mathrm{On}^A$ and $h(a) = 0$ for some $a \in A$, then $\Pi h = \emptyset$. So, to avoid triviality $h(a) > 0$ is assumed for all $a \in A$, whenever the expression Πh is used. Hence if I is an ideal over A then every $g \in \mathrm{On}^A$ such that $g <_I h$ is $=_I$ equivalent to some function in Πh. In fact, we shall usually consider reduced products $\Pi h / I$ for functions h such that $h(a) > 0$ is always a limit ordinal, and hence every function in Πh is $<$-bounded (everywhere dominated) by some function in Πh.

Suppose that F is a (non-empty) set of functions in On^A such that for every $f \in F$ there exists some $f' \in F$ with $f <_I f'$. Then $h \in \mathrm{On}^A$ is an exact upper bound of F if h is a least upper bound of F and for every $g <_I h$ there is some $f \in F$ with $g <_I f$ (namely F is cofinal in the lower $<_I$ cone determined by h). Actually it is not necessary to require that h is a least upper bound of F since this follows from the assumptions that h is an upper bound of F and F is cofinal in $\Pi h / I$. Thus if $F \subseteq \Pi h / I$ then h is an exact upper bound of F iff F is cofinal in $\Pi h / I$.

If h is an exact upper bound of F and $A_0 \in I^+$ then $h \restriction A_0$ is an exact upper bound of $\langle f \restriction A_0 \mid f \in F \rangle$ with respect to the proper ideal $I \cap \mathcal{P}(A_0)$.

If h is an exact upper bound of F with respect to some ideal I over A and $J \supseteq I$ is a larger ideal over A, then h is an exact upper bound of F modulo J as well.

The definition of "true cofinality" of a reduced product is so important for the pcf theory that we restate it for this case.

2.2 Definition. We say that $\mathrm{tcf}(\Pi h / I) = \lambda$ iff λ is a regular cardinal and there exists a $<_I$-increasing sequence $f = \langle f_\xi \mid \xi < \lambda \rangle$ in Πh that is cofinal in $\Pi h / I$.

Projections

We shall often encounter the following situation.

1. A is a non-empty set of indices, and $S = \langle S_a \mid a \in A \rangle$ is a sequence of sets of ordinals. The sup_of_S function is defined on A by taking $a \in A$ to $\sup S_a$.

2. An ordinal function $f \in \mathrm{On}^A$ is given that is bounded by the sup_of_S, namely $f(a) < \sup S_a$ for every $a \in A$.

Then we define the *projection* of f onto ΠS, denoted $\mathrm{proj}(f, S)$, as the function $f^+ \in \Pi S$ defined by

$$f^+(a) = \min(S_a \setminus f(a)).$$

So $f^+(a) = f(a)$ in case $f(a) \in S_a$, and otherwise $f^+(a)$ is the least ordinal in S_a above $f(a)$. (There is such an ordinal by our assumption.) It is clear that f^+ is the least function in ΠS that bounds f, and that $f_1 \leq f_2$ implies $f_1^+ \leq f_2^+$.

We shall apply projections in the presence of an ideal I over A. If $f \in \mathrm{On}^A$ is any function, not necessarily bounded by sup_of_S, we define $f^+ = \mathrm{proj}(f, S)$ as follows. For $a \in A$ such that $f(a) < \sup S_a$, we define $f^+(a) = \min(S_a \setminus f(a))$ as before, and for $a \in A$ such that $f(a) \geq \sup S_a$ we define $f^+(a) = 0$. Clearly, $f_1 =_I f_2$ implies that $f_1^+ =_I f_2^+$. It follows, in case $f <_I \sup S$, that f^+ is the $\leq_I$-least function in $\Pi_{a\in A}\ S_a$ that $\leq_I$-bounds f, up to $=_I$ equivalence.

Given an ideal I over a set A and an ordinal function $h \in \mathrm{On}^A$, we are interested in the existence and value of the true cofinality of $\Pi h/I$. Our first step is to reduce this question to ultraproducts of regular cardinals, and we can proceed as follows. Choose for every $a \in A$ a cofinal set $S(a) \subseteq h(a)$ of order-type $\mathrm{cf}(h(a))$. By our assumption that $h(a) > 0$ is always a limit, non-zero ordinal, the order type of $S(a)$ is a regular infinite cardinal. Then the collections Πh and $\Pi_{a\in A}\ S(a)$ are cofinally equivalent. That is for every $f \in \Pi h$ there is $g \in \Pi S$ with $f \leq g$ (namely it projection), and vice versa.

Next, $\Pi_{a\in A}\ S(a)$ is isomorphic to $\Pi_{a\in A}\ |S(a)| = \Pi_{a\in A}\mathrm{cf}(h(a))$. This is also the case when an ideal I over A is introduced and the relation $\leq_I$ is considered. Then $\Pi h/I$ has the same cofinality and true cofinality as $\Pi_{a\in A}\ \mathrm{cf}(h(a))/I$. Hence it suffices to consider reduced products $\Pi_{a\in A}\ k(a)/I$ of functions k such that $k(a)$ are infinite regular cardinals. As the following lemma shows, in some cases we may even take k to be one-to-one.

2.3 Lemma. *Suppose that $c : A \to$ Regular_Cardinals is a function and $B = \{c(a) \mid a \in A\}$ is its range. Suppose I is any ideal over A, and J is its Rudin-Keisler projection on B defined by*

$$X \in J \textit{ iff } X \subseteq B \textit{ and } c^{-1}X \in I,$$

where $c^{-1}X = \{a \in A \mid c(a) \in X\}$. Then there is an order-preserving isomorphism $h : \Pi B/J \to \Pi_{a\in A}\ c(a)/I$ defined by $h([e]_J) = [e \circ c]_I$, for every $e \in \Pi B$. If $|A| < \min B$, then

$$\mathrm{tcf}(\Pi B/J) = \mathrm{tcf}(\Pi_{a\in A}\ c(a)/I) \tag{I.4}$$

in the sense that existence of the true cofinality for one of $\Pi B/J$ *and* $\Pi c/I$ *implies existence for the other poset as well, and these cofinalities are equal.*

Proof. For every $e \in \Pi B$ define $\bar{e} \in \Pi c$ by $\bar{e}(a) = e(c(a))$. That is, $\bar{e} = e \circ c$. Then $e_1 =_J e_2$ iff $\bar{e}_1 =_I \bar{e}_2$, and $e_1 <_J e_2$ iff $\bar{e}_1 <_I \bar{e}_2$. Thus $h([e]_J) = \bar{e}/I$ induces an isomorphism from $\Pi B/J$ into $\Pi c/I$. Hence tcf $(\Pi B/J)$ is the same as the true cofinality of

$$G = \{h([e]_J) \mid e \in \Pi B\}$$

in $<_I$. If $|A| < \min B$, then G will be shown to be cofinal in $\Pi c/I$ and this implies (I.4). (In general, if G is any cofinal subset of a partial ordering $(P, <_P)$, then G and P have the same true cofinality.)

Now G is cofinal in $\Pi c/I$, because any $g \in \Pi c$ is bounded by $\bar{f}$ where $f \in \Pi B$ is defined by

$$f(b) = \sup\{g(a) \mid a \in A \text{ and } c(a) = b\}.$$

The fact that $|A| < b$ is used here to deduce that this supremum is below the regular cardinal $b \in B$, and hence that $f \in \Pi B$. Since $\bar{f}/I \in G$, G is cofinal in Πc. ⊣

To see how this lemma is applied, suppose that λ is a regular cardinal and $f = \langle f_\xi \mid \xi < \lambda \rangle$ is a $<_I$ increasing sequence of functions $f_\xi \in \mathrm{On}^A$. Then (as we have said) $h \in \mathrm{On}^A$ is an exact upper bound of f iff f is cofinal in $\Pi h/I$. In this case it follows that $\mathrm{tcf}(\Pi h/I) = \lambda$ and hence that the true cofinality of $\Pi_{a \in A}\ \mathrm{cf}(h(a))$ is λ. Let $B = \{\mathrm{cf}(h(a)) \mid a \in A\}$ be the set of cofinalities of the range of h. The preceding lemma shows that λ is the true cofinality of a reduced product of B, if $|A| < \mathrm{cf}(h(a))$ for every $a \in A$.

2.1. Existence of exact upper bounds

An important piece of the pcf theory is the determination of conditions that ensure the existence of exact upper bounds. Recall that an exact upper bound of a $<_I$-increasing sequence $\langle f_\xi \mid \xi < \lambda \rangle$ of functions in On^A is a function $g \in \mathrm{On}^A$ that bounds every f_ξ in the $\leq_I$ relation and satisfies the additional requirement that if $d <_I g$ then $d <_I f_\xi$ for some $\xi < \lambda$. The following and Definition 2.8 are central in our presentation of the pcf theory.

2.4 Definition. [*Strongly increasing*] Suppose that I is an ideal over A and $f = \langle f_\xi \mid \xi \in L \rangle$ is a $<_I$-increasing sequence of functions $f_\xi \in \mathrm{On}^A$, where L is a set of ordinals. Then f is said to be *strongly increasing* if there are null sets $Z_\xi \in I$, for $\xi \in L$, such that whenever $\xi_1 < \xi_2$ are in L

$$a \in A \setminus (Z_{\xi_1} \cup Z_{\xi_2}) \implies f_{\xi_1}(a) < f_{\xi_2}(a).$$

2.5 Exercise. An even stronger property would be to require that there are null sets $Z_\xi \in I$ for $\xi \in L$ such that whenever $\xi_1 < \xi_2$

$$a \in A \setminus Z_{\xi_2} \to f_{\xi_1}(a) < f_{\xi_2}(a).$$

Prove that a sequence $f = \langle f_\xi \mid \xi \in L \rangle$ satisfies this stronger property iff for every $\xi \in L$

$$\sup\{f_\alpha + 1 \mid \alpha \in L \cap \xi\} \leq_I f_\xi. \tag{I.5}$$

(Recall that $f + 1$ is the function that takes x to $f(x) + 1$.)

2.6 Exercise. Let I be an ideal over A, $\lambda > |A|$ be a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda \rangle$ be a $<_I$ increasing sequence of functions in On^A. Then the following conditions are equivalent:

1. f contains a strongly increasing subsequence of length λ.
2. f has an exact upper bound h such that $\mathrm{cf}(h(a)) = \lambda$ for (I-almost) all $a \in A$.
3. f is cofinally equivalent to some $<$ (i.e. everywhere) increasing sequence of length λ.

Hint. If f (or a subsequence) is strongly increasing, let $Z_\xi \in I$ be the null sets associated with f_ξ and define

$$h(a) = \sup\{f_\xi(a) \mid a \notin Z_\xi\}.$$

Prove that h is an exact upper bound as required to prove that 1 implies 2.

Since $|A| < \lambda$, it is obvious that 2 implies 3. (For every $a \in A$ choose a cofinal subset of $h(a)$ of order-type λ, and let d_ξ be the "flat" function which assigns to $d_\xi(a)$ the ξth point in the $h(a)$ cofinal subset.)

To prove that 3 implies 1, use the following lemma.

2.7 Lemma. *(The sandwich argument) Suppose that $d = \langle d_\xi \mid \xi \in \lambda \rangle$ is strongly increasing and $f_\xi \in \mathrm{On}^A$ is such that*

$$d_\xi <_I f_\xi \leq_I d_{\xi+1} \textit{ for every } \xi \in \lambda.$$

Then $\langle f_\xi \mid \xi \in \lambda \rangle$ is also strongly increasing.

Proof. Let $Z_\xi \in I$ be the null sets that affirm that the sequence d is strongly increasing. For every f_ξ, sandwiched between d_ξ and $d_{\xi+1}$, there exists $W_\xi \in I$ such that

$$d_\xi(a) < f_\xi(a) \leq d_{\xi+1}(a) \text{ for all } a \in A \setminus W_\xi.$$

Define $Z^\xi = W_\xi \cup Z_\xi \cup Z_{\xi+1}$. Then $Z^\xi \in I$, and if $\xi_1 < \xi_2$ then for every $a \in A \setminus (Z^{\xi_1} \cup Z^{\xi_2})$

$$f_{\xi_1}(a) \leq d_{\xi_1+1}(a) \leq d_{\xi_2}(a) < f_{\xi_2}(a).$$

$\dashv$

2.8 Definition. Suppose that I is an ideal over a set A, λ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda \rangle$ is a $<_I$-increasing sequence of functions $f_\xi \in \mathrm{On}^A$. For any regular cardinal κ such that $\kappa \leq \lambda$ the following crucial property of κ (and f etc.) is denoted $(*)_\kappa$:

$(*)_\kappa$ Whenever $X \subseteq \lambda$ is unbounded, then for some $X_0 \subseteq X$ of order type κ, $\langle f_\xi \mid \xi \in X_0 \rangle$ is strongly increasing.

Thus $(*)_\kappa$ is some kind of a partition relation, saying that any unbounded subsequence $\langle f_\xi \mid \xi \in X \rangle$ contains a strongly increasing subsequence of length κ. Clearly $(*)_\kappa$ implies $(*)_{\kappa'}$ for all regular $\kappa' < \kappa$.

2.9 Exercise. 1. Assume $\kappa < \lambda$. Prove that $(*)_\kappa$ holds iff the set of ordinals $\delta \in \lambda$ with $\mathrm{cf}(\delta) = \kappa$ and such that $\langle f_\xi \mid \xi \in X_0 \rangle$ is strongly increasing for some unbounded set $X_0 \subseteq \delta$ is stationary in λ.

2. Use the Erdos-Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ to prove that if $\lambda \geq (2^{|A|})^+$ and f is a $<_I$ increasing sequence of functions as above, of length λ, then $(*)_{|A|^+}$ holds.

Hint for 2. For $i < j$, if there exists some $a \in A$ such that $f_i(a) > f_j(a)$, then define $c(i,j) = a$ for such an a. Otherwise define $c(i,j) = -1$. The homogeneous set must be of color -1, and $(*)_{|A|^+}$ can be derived by taking a subsequence.

We shall give (in Lemma 2.19) conditions that ensure property $(*)_\kappa$ (without any assumptions on 2^κ), but meanwhile the following lemma and theorem explain the main use of $(*)_\kappa$.

2.10 Definition (Bounding projection). Suppose that I is an ideal over A, λ is a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in On^A. Let $\kappa \leq \lambda$ be any regular cardinal. We say that f has the *bounding projection property* for κ if whenever $S = \langle S(a) \mid a \in A \rangle$ with $S(a) \subset \mathrm{On}$ and $|S(a)| < \kappa$ is such that the sequence f is $<_I$-bounded by the function sup_of_S, then there exists $\xi < \lambda$ such that $f^+_\xi = \mathrm{proj}(f_\xi, \langle S_a \mid a \in A \rangle)$ is an upper bound of f in the $<_I$ relation. (Recall that sup_of_$S(a) = \sup S(a)$ for all $a \in A$.)

2.11 Exercise. 1. If $f = \langle f_\xi \mid \xi < \lambda \rangle$ has the bounding projection property for κ and $f' = \langle f'_\xi \mid \xi < \lambda \rangle$ is such that $f'_\xi =_I f_\xi$ for every ξ, then f' too has the bounding projection property for κ.

2. A seemingly weaker property is obtained by requiring that the sup_of_S map $<$-bounds (i.e. everywhere) each f_ξ. Prove that these two definitions are equivalent.

2.12 Lemma (The bounding projection lemma). *Suppose that I is an ideal over A, $\lambda > |A|$ is a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda\rangle$ is a $<_I$-increasing sequence satisfying $(*)_\kappa$ for a regular cardinal κ such that $|A| < \kappa \le \lambda$. Then f satisfies the bounding projection property for κ.*

Later on, we shall see that $(*)_\kappa$ is, in fact, equivalent *to the bounding projection property for κ (see Theorem 2.15 for an exact formulation).*

Proof. Suppose that the lemma is false and S is a counter-example, and we shall obtain a contradiction. By changing each f_ξ on an I set, we do not spoil the $(*)_\kappa$ property, and we may assume that $f_\xi(a) < \sup S_a$ for all $a \in A$ (here $S_a = S(a)$). Then define

$$f_\xi^+ = \operatorname{proj}(f_\xi, \langle S_a \mid a \in A\rangle).$$

Since f_ξ^+ is not a $<_I$-upper bound, there exists $\xi' < \lambda$ such that $\le(f_\xi^+, f_{\xi'}) \in I^+$. That is $f_\xi^+(a) \le f_{\xi'}(a)$ for an I-positive set of $a \in A$. Hence $<(f_\xi^+, f_{\xi''}) \in I^+$ for every ξ'' above ξ'. This enables the definition of an unbounded set $X \subseteq \lambda$ such that

$$\text{if } \xi, \xi' \in X \text{ and } \xi < \xi' \text{ then } <(f_\xi^+, f_{\xi'}) \in I^+.$$

Since $(*)_\kappa$ holds, there exists a set $X_0 \subseteq X$ of order-type κ such that $\langle f_\xi \mid \xi \in X_0\rangle$ is strongly increasing. Let $Z_\xi \in I$ for $\xi \in X_0$ be as in the definition of strong increase (2.4).

For every $\xi \in X_0$ let $\xi' = \min X_0 \setminus (\xi + 1)$ be the successor of ξ in X_0, and pick

$$a_\xi \in <(f_\xi^+, f_{\xi'}) \setminus (Z_\xi \cup Z_{\xi'}).$$

As $\kappa > |A|$, we may find a single $a \in A$ such that $a = a_\xi$ for a subset X_1 of X_0 of cardinality κ. Now for $\xi_1 < \xi_2$ in X_1

$$f_{\xi_1}^+(a) < f_{\xi_1'}(a) \le f_{\xi_2}(a) \le f_{\xi_2}^+(a).$$

(The first inequality is a consequence of $a_{\xi_1} \in <(f_{\xi_1}^+, f_{\xi_1'})$, the second follows from $\xi_1' \le \xi_2$ and the fact that

$$a = a_{\xi_1} = a_{\xi_2} \in A \setminus (Z_{\xi_1'} \cup Z_{\xi_2}),$$

and the third inequality is obvious from the definition of $f_{\xi_2}^+$.)

But now $f_\xi^+(a) \in S_a$ turns out to be strictly increasing with $\xi \in X_1$, which is absurd since $|S_a| < \kappa$. $\dashv$

2.13 Theorem (Exact upper bounds). *Suppose that I is an ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in* On^A *that satisfies the bounding projection property for $|A|^+$. Then f has an exact upper bound.*

Proof. Assume the $|A|^+$ bounding projection property for a sequence f that is $<_I$-increasing of length a regular cardinal $\lambda > |A|^+$. We shall prove first that there exists a minimal upper bound to f, and then prove that this bound is necessarily an exact upper bound. Seeking a contradiction, suppose that f has no minimal upper bound. So for every $h \in \mathrm{On}^A$, if h is an upper bound to the sequence f then it is not a minimal upper bound, and there is another upper bound $h' \in \mathrm{On}^A$ to f such that $h' \leq h$ and $<(h', h) \in I^+$.

We shall define by induction on $\alpha < |A|^+$ a sequence $S^\alpha = \langle S^\alpha(a) \mid a \in A\rangle$ of sets of ordinals satisfying $|S^\alpha(a)| \leq |A|$, and such that:

1. The sequence of functions f is bounded by the map $a \mapsto \sup S^\alpha(a)$. So, the projections can always be defined.

2. The sets $S^\alpha(a)$ are increasing with α: if $\alpha < \beta$ then $S^\alpha(a) \subseteq S^\beta(a)$ for every $a \in A$. For a limit ordinal δ, $S^\delta(a) = \bigcup_{\alpha<\delta} S^\alpha(a)$.

To define S^0, we pick a function h_0 that bounds f and define $S^0(a) = \{h_0(a)\}$.

Suppose that $S^\alpha = \langle S^\alpha(a) \mid a \in A\rangle$ has been defined. Since the bounding projection property for $|A|^+$ holds and the cardinality of $S^\alpha(a)$ is $\leq |A|$, there exists some $\xi = \xi(\alpha) < \lambda$ such that $h_\alpha = \mathrm{proj}(f_\xi, S^\alpha)$ is an upper bound of f. It follows for every ξ' satisfying $\xi \leq \xi' < \lambda$ that $h_\alpha =_I \mathrm{proj}(f_{\xi'}, S^\alpha)$.

Since h_α is not a minimal upper bound, there exists an upper bound u to the sequence f such that $u \leq h_\alpha$ and

$$<(u, h_\alpha) \in I^+.$$

Define $S^{\alpha+1}(a) = S^\alpha(a) \cup \{u(a)\}$. Then $\mathrm{proj}(f_\xi, S^{\alpha+1}) =_I u$ for all $\xi(\alpha) \leq \xi < \lambda$.

Now let $\xi < \lambda$ be a fixed ordinal greater than every $\xi(\alpha)$ for $\alpha < |A|^+$ (recall that λ is a regular cardinal above $|A|^+$). Consider the functions $H_\alpha = \mathrm{proj}(f_\xi, S^\alpha)$ for $\alpha < |A|^+$. Since f_ξ is above $f_{\xi(\alpha)}$, $H_\alpha =_I h_\alpha$. Thus $<(H_{\alpha+1}, H_\alpha) \in I^+$. Since $\alpha_1 < \alpha_2 < |A|^+$ implies that $S^{\alpha_1}(a) \subseteq S^{\alpha_2}(a)$ for all $a \in A$, the sequence of projections $\langle H_\alpha \mid \alpha < |A|^+\rangle$ thus obtained satisfies the following property:

$$\text{If } \alpha_1 < \alpha_2 < |A|^+, \text{ then } H_{\alpha_2} \leq H_{\alpha_1} \text{ and } <(H_{\alpha_2}, H_{\alpha_1}) \in I^+.$$

Yet this is impossible and leads immediately to a contradiction. For every $\alpha < |A|^+$ pick some $a \in A$ such that $H_{\alpha+1}(a) < H_\alpha(a)$. Then the same fixed $a \in A$ is picked for an unbounded set of indices $\alpha \in |A|^+$. Yet as the functions H_α are $\leq$-decreasing, this yields an infinite strictly descending sequence of ordinals! $\dashv$

Now that the existence of a minimal upper-bound is established, the following lemma concludes the theorem.

2.14 Lemma. *Suppose that I is an ideal over A, λ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in On^A that satisfies the bounding projection property for $\kappa = 3$. Let h be a minimal upper bound of f. Then h is an exact upper bound.*

Proof. Assume that f satisfies the bounding projection property for 3, and h is a minimal upper bound of f. Suppose that $g \in \mathrm{On}^A$ is such that $g <_I h$. We must find f_ξ in the sequence f with $g <_I f_\xi$. For simplicity, and without loss of generality, we can assume that $g(a) < h(a)$ for all $a \in A$.

Define $S_a = \{g(a), h(a)\}$ for every $a \in A$. The bounding projection property implies the existence of $\xi < \lambda$ for which $f_\xi^+ = \mathrm{proj}(f_\xi, \langle S_a \mid a \in A\rangle)$ is an upper bound of the sequence f. We shall prove that $g <_I f_\xi$ as required. Observe that

$$f_\xi^+ =_I h \tag{I.6}$$

or else $f_\xi^+(a) = g(a) < h(a)$ for a positive set of a's in A. But then f_ξ^+ is an upper-bound of f that is smaller than the minimal upper bound h on a positive set of indices, and this is impossible. Hence (I.6). Yet, for every a such that $f_\xi^+(a) = h(a)$, $g(a) < f_\xi(a)$ follows from the fact that $g(a) \in S_a$. Thus $g <_I f_\xi$. This proves the lemma. $\dashv$

The bounding projection lemma 2.12 and the exact upper bounds theorem 2.13 show together that a $<_I$-increasing sequence of length a regular cardinal $\lambda > |A|^+$ and which satisfies $(*)_{|A|^+}$ has necessarily an exact upper bound h. As we shall see in the following theorem it can be deduced that

$$\forall a \in A \ \mathrm{cf}(h(a)) \geq |A|^+.$$

2.15 Theorem. *Suppose that I is an ideal over A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a $<_I$-increasing sequence of functions in On^A. Then for every regular cardinal κ such that $|A|^+ \leq \kappa \leq \lambda$ the following are equivalent.*

1. *$(*)_\kappa$ holds for f.*
2. *f satisfies the bounding projection property for κ.*

3. *The sequence f has an exact upper bound g for which*
$$\{a \in A \mid \mathrm{cf}(g(a)) < \kappa\} \in I.$$

Proof. Let κ be a regular cardinal such that $|A|^+ \leq \kappa \leq \lambda$. Implication *1* $\implies$ *2* was proved in Lemma 2.12, and so we next establish *2* $\implies$ *3*.

Since f satisfies the bounding projection property for some cardinal that is $\geq |A|^+$, it satisfies the bounding projection property for $|A|^+$. Theorem 2.13 above implies that f has an exact upper bound g. This exact upper bound is determined up to $=_I$, and we may assume that $g(a)$ is never 0 or a successor ordinal (recall that the sequence f is $<_I$-increasing).

Suppose that $P = \{a \in A \mid \mathrm{cf}(g(a)) < \kappa\} \in I^+$, in contradiction to *3*. Choose, for every $a \in P$, $S(a) \subseteq g(a)$ cofinal in $g(a)$ and such that order-type$(S(a)) < \kappa$. For $a \in A \setminus P$ define $S(a) = \{g(a)\}$. Then the bounding projection property for κ gives some $\xi < \lambda$ such that the projection
$$f^+_\xi = \mathrm{proj}(f_\xi, \langle S(a) \mid a \in A\rangle)$$
is an upper bound of f in $\Pi_{a\in A}\, S(a)$. But this is impossible since $f^+_\xi \restriction P < g \restriction P$ (everywhere on P) is in contradiction to our assumption that g is the $\leq_I$-minimal upper bound of f.

We now proceed with *3* $\implies$ *1*. Suppose that g is an exact upper bound for f such that $\mathrm{cf}(g(a)) \geq \kappa$ for all $a \in A$ (change g on a null set if necessary). Choose $S(a) \subseteq g(a)$ cofinal in $g(a)$, closed, and with order type $\mathrm{cf}(g(a))$. So order-type$(S(a)) \geq \kappa$. We prove that $(*)_\kappa$ holds. Assuming that $X \subseteq \lambda$ is unbounded, we shall find $X_0 \subseteq X$ of order-type κ over which f is strongly increasing. For this we intend to define by induction on $\alpha < \kappa$ a function $h_\alpha \in \Pi_{a\in A}\, S(a) = \Pi S$ and an index $\xi(\alpha) \in X$ such that

1. $h_\alpha <_I f_{\xi(\alpha)} <_I h_{\alpha+1}$.
2. The sequence $\langle h_\alpha \mid \alpha < \kappa\rangle$ is $<$ increasing (increasing everywhere). And hence it is certainly strongly increasing.

Then the sandwich argument (Lemma 2.7) will show that $\{f_{\xi(\alpha)} \mid \alpha < \kappa\}$ is strongly increasing.

The functions h_α are defined as follows.

1. $h_0 \in \Pi_{a\in A}\, S(a)$ is any function.
2. If $\delta < \kappa$ is a limit ordinal, then define
$$h_\delta = \sup\{h_\alpha \mid \alpha < \delta\}.$$
That is
$$h_\delta(a) = \bigcup\{h_\alpha(a) \mid \alpha < \delta\}$$
for every $a \in A$. Since each $S(a)$ has regular order type $\geq \kappa$, and as $\delta < \kappa$, clearly $h_\delta \in \Pi_{a\in A}\, S(a)$.

3. If $h_\alpha \in \Pi_{a\in A}\ S(a)$ is defined then it is bounded by g (since $S(a) \subseteq g(a)$) and hence (as g is an exact upper bound) $h_\alpha <_I f_\xi$ for some $\xi \in X$, which we denote $\xi(\alpha)$. Now let $f^+_{\xi(\alpha)} = \text{proj}(f_{\xi(\alpha)}, S)$ be the projection function, and define $h_{\alpha+1} \in \Pi S$ so that $h_{\alpha+1} > \sup\{h_\alpha, f^+_{\xi(\alpha)}\}$.

Thus $h_{\alpha+1} > h_\alpha$ and since $f_{\xi(\alpha)} \leq_I f^+_{\xi(\alpha)}$ we have

$$h_\alpha <_I f_{\xi(\alpha)} <_I h_{\alpha+1}, \text{ for every } \alpha. \tag{I.7}$$

Hence

$$X_0 = \{\xi(\alpha) \mid \alpha \in \kappa\} \subseteq X$$

is an increasing enumeration, and it is an evidence for $(*)_\kappa$ (by the sandwich argument and since $\langle h_\alpha \mid \alpha < \kappa\rangle$ is strongly increasing). ⊣

We shall give in Lemma 2.19 below a useful condition on f from which $(*)_\kappa$ follows. But first we need a combinatorial theorem.

2.16 Definition. If $S \subseteq \lambda$ is a stationary set, then a club guessing sequence is a sequence $\langle C_\delta \mid \delta \in S\rangle$, where each $C_\delta \subseteq \delta$ is closed unbounded in δ, such that for every closed unbounded $D \subseteq \lambda$ there exists some $\delta \in S$ with $C_\delta \subseteq D$.

We shall use the notation $S^\lambda_\kappa = \{\delta \in \lambda \mid \text{cf}(\delta) = \kappa\}$. Clearly for regular infinite cardinals $\kappa < \lambda$, S^λ_κ is stationary in λ.

2.17 Theorem (Club Guessing). *For every regular cardinal κ, if λ is a cardinal such that $\text{cf}(\lambda) \geq \kappa^{++}$, then any stationary set $S \subseteq S^\lambda_\kappa$ has a club-guessing sequence $\langle C_\delta \mid \delta \in S\rangle$ (such that $C_\delta \subseteq \delta$ is closed unbounded of order type κ).*

Proof. We shall prove this for uncountable κ's, though the theorem holds for $\kappa = \aleph_0$ as well.

Let $S \subseteq S^\lambda_\kappa$ be any stationary set. Fix a sequence $C = \langle C_\delta \mid \delta \in S\rangle$ such that $C_\delta \subseteq \delta$ is closed unbounded of order type κ, for every $\delta \in S$. If $E \subseteq \lambda$ is any closed unbounded set, define

$$C \mid E = \langle C_\delta \cap E \mid \delta \in S \cap E'\rangle.$$

Here $E' = \{\delta \in E \mid E \cap \delta \text{ is unbounded in } \delta\}$ is the set of accumulation points of E. Clearly $E' \subseteq E$ is closed unbounded. The sequence $C \mid E$ is defined on $S \cap E'$ in order to ensure that $C_\delta \cap E$ is closed unbounded in δ.

We claim that for some closed unbounded set $E \subseteq \lambda$, $C \mid E$ is club-guessing. (The theorem demands a sequence defined on *every* $\delta \in S$, but this is trivially obtained once a guessing sequence is defined on a closed unbounded set intersected with S.)

To prove this claim, assume that it is false, and for every closed unbounded set $E \subseteq \lambda$ there is some closed unbounded set $D_E \subseteq \lambda$ not guessed by $C \mid E$. That is, for every $\delta \in S \cap E'$

$$C_\delta \cap E \nsubseteq D_E.$$

So we can define a decreasing (under inclusion) sequence of closed unbounded sets $E^\alpha \subseteq \lambda$ for $\alpha < \kappa^+$ by induction on α as follows.

1. $E^0 = \lambda$.

2. If $\gamma < \kappa^+$ is a limit ordinal, and E^α for $\alpha < \gamma$ are already defined, let $E^\gamma = \bigcap\{E^\alpha \mid \alpha < \gamma\}$. Clearly $E^\gamma \subseteq \lambda$ is closed unbounded.

3. If E^α is defined, then $E^{\alpha+1} = (E^\alpha \cap D_{E^\alpha})'$. So for every $\delta \in S \cap E^{\alpha+1}$, $C_\delta \cap E^\alpha \nsubseteq E^{\alpha+1}$.

Let $E = \bigcap\{E^\alpha \mid \alpha < \kappa^+\}$. Again $E \subseteq \lambda$ is closed unbounded because $\mathrm{cf}(\lambda) > \kappa^+$.

Now we get the contradiction. Take any $\delta \in S \cap E$. There exists some $\alpha < \kappa^+$ such that $C_\delta \cap E = C_\delta \cap E^\alpha$ (since the sets E^α are decreasing in $\subseteq$ and C_δ has cardinality κ). So $C_\delta \cap E^\alpha = C_\delta \cap E^{\alpha'}$ for every $\alpha' > \alpha$, and in particular for $\alpha' = \alpha + 1$. But as $\delta \in S \cap E^{\alpha+1}$, $C_\delta \cap E^\alpha \nsubseteq E^{\alpha+1}$. $\dashv$

2.18 Exercise. 1. Club guessing is a relative of the diamond principle which gives much stronger guessing properties. For example, prove that $\diamondsuit^+_{\omega_2}$ implies a sequence $\langle C_\delta \mid \delta \in S^{\omega_2}_{\omega_1}\rangle$ with C_δ closed unbounded in δ such that, for every closed unbounded set $E \subseteq \omega_2$, there exists a closed unbounded set $D \subseteq \omega_2$ such that for every $\delta \in S^{\omega_2}_{\omega_1} \cap D$, C_δ is almost contained in E (i.e. except a bounded set). Prove that it is not possible to have full guessing at a closed unbounded set. That is, it is not possible to require that $C_\delta \subseteq E$ for every $\delta \in S^{\omega_2}_{\omega_1} \cap D$.

2. Prove the club-guessing theorem for $\kappa = \aleph_0$ as well.

   Hint. For $S \subseteq S^\lambda_{\aleph_0}$ fix $C = \langle C_\delta \mid \delta \in S\rangle$ where each C_δ is an ω-sequence unbounded in δ. For every closed unbounded set $E \subseteq \lambda$ define the "gluing to E" sequence $C \mid E = \langle C^*_\delta \mid \delta \in S \cap E^*\rangle$ by

   $$C^*_\delta(n) = \max(E \cap (C_\delta(n) + 1)).$$

   Try to prove that for some club $E \subseteq \lambda$, $C \mid E$ is club guessing. Have enough patience for ω_1 trials.

Club guessing is used in the following lemma which produces sequences that satisfy $(*)_\kappa$.

2.19 Lemma. *Suppose that*

1. *I is a proper ideal over A.*

2. *κ and λ are regular cardinals such that $\kappa^{++} < \lambda$.*

3. *$f = \langle f_\xi \mid \xi < \lambda \rangle$ is a sequence of length λ of functions in On^A that is $<_I$-increasing and satisfies the following requirement:*

   For every $\delta < \lambda$ with $\mathrm{cf}(\delta) = \kappa^{++}$ there is a closed unbounded set $E_\delta \subseteq \delta$ such that for some $\delta' \geq \delta$ in λ

$$\sup\{f_\alpha \mid \alpha \in E_\delta\} <_I f_{\delta'}. \tag{I.8}$$

Then $(*)_\kappa$ holds for f.

Proof. Let $S = S_\kappa^{\kappa^{++}}$ be the stationary subset of κ^{++} consisting of all ordinals with cofinality κ. Fix a club-guessing sequence on S: $\langle C_\alpha \mid \alpha \in S \rangle$. So for every $\alpha \in S$, $C_\alpha \subseteq \alpha$ is closed unbounded, of order type κ, and for every closed unbounded set $C \subseteq \kappa^{++}$ there is $\delta \in S$ such that $C_\delta \subseteq C$.

Now let $U \subseteq \lambda$ be an unbounded set, and we shall find $X_0 \subseteq U$ of order type κ such that $\langle f_\xi \mid \xi \in X_0 \rangle$ is strongly increasing. For this we first define an increasing and continuous sequence $\langle \xi(i) \mid i < \kappa^{++} \rangle \subset \lambda$ of order-type κ^{++} by the following recursive procedure.

We start with an arbitrary $\xi(0)$. For i limit, $\xi(i) = \sup\{\xi(k) \mid k < i\}$.

Suppose for some $i < \kappa^{++}$ that $\{\xi(k) \mid k \leq i\}$ has been defined.

For every $\alpha \in S$ define

$$h_\alpha = \sup\{f_{\xi(k)} \mid k \leq i \wedge k \in C_\alpha\}. \tag{I.9}$$

Then ask: is there an ordinal $\sigma > \xi(i)$ below λ such that $h_\alpha <_I f_\sigma$? If the answer is positive, let σ_α be the least such $\sigma < \lambda$, and, if negative, let σ_α be $\xi(i) + 1$.

Since $\lambda > \kappa^{++}$ is regular, we can define

$$\xi(i+1) > \sup\{\sigma_\alpha \mid \alpha \in S\} \;\text{ with } \xi(i+1) \in U.$$

It follows, in case the answer for h_α is positive, that

$$h_\alpha <_I f_{\xi(i+1)}.$$

Finally $D = \{\xi(k) \mid k \in \kappa^{++}\}$ is closed and has order type κ^{++}. Let $\delta = \sup D$. Then D is closed unbounded in $\delta < \lambda$, and $\mathrm{cf}(\delta) = \kappa^{++}$. By assumption there is a closed unbounded set $E_\delta \subseteq \delta$ such that (I.8) holds. Thus for some $f_{\delta'}$

$$\sup\{f_\xi \mid \xi \in E_\delta\} <_I f_{\delta'}. \tag{I.10}$$

Observe that $D \cap E_\delta$ is closed unbounded in δ, and thus $C = \{i \in \kappa^{++} \mid \xi(i) \in E_\delta\}$ is closed unbounded. Hence for some $\alpha \in S$, $C_\alpha \subseteq C$. So (I.10) implies that

$$\sup\{f_{\xi(i)} \mid i \in C_\alpha\} <_I f_{\delta'}. \tag{I.11}$$

Let $N_\alpha \subseteq C_\alpha$ be the set of non-accumulation points of C_α, that is those $i \in C_\alpha$ for which $C_\alpha \cap i$ is bounded in i. We shall prove that $\{f_{\xi(i)} \mid i \in N_\alpha\}$ is strongly increasing. Since $\xi(i+1) \in U$ for every i, the sandwich lemma (2.7) gives a strongly increasing subsequence of $\{f_\alpha \mid \alpha \in U\}$ of order-type κ.

Claim. *For every $i < j$ both in C_α*

$$\sup\{f_{\xi(k)} \mid k \leq i \wedge k \in C_\alpha\} <_I f_{\xi(j)}. \tag{I.12}$$

Proof of the claim. Recall how $f_{\xi(i+1)}$ was defined. We considered (I.9) and asked if h_α is $<_I$ dominated by some f_σ. The answer was positive, since $f_{\delta'}$ is such a bound. Hence the claim and the lemma follow. ⊣

2.20 Exercise. Let κ and λ be regular cardinals with $\kappa^{++} < \lambda$, and let F be any function with $\mathrm{dom}(F) \subseteq [\lambda]^{<\kappa}$ and such that $F(X) \in \lambda$ for $X \in \mathrm{dom}(F)$. Suppose that for every $\delta \in S^\lambda_{\kappa^{++}}$ there exists a closed unbounded set $E_\delta \subseteq \delta$ such that $[E_\delta]^{<\kappa} \subseteq \mathrm{dom}(F)$. Then the following set S is stationary: the set of all ordinal $\alpha \in S^\lambda_\kappa$ for which there exists a closed unbounded set $D \subseteq \alpha$ with the property that, for any $a < b$ both in D, $F(\{d \in D \mid d \leq a\}) < b$.

A typical application of Lemma 2.19 is the following

2.21 Theorem. *Suppose that I is a proper ideal over a set of regular cardinals A, and λ is a regular cardinal such that $\Pi A/I$ is λ-directed. If $\langle g_\xi \mid \xi < \lambda\rangle$ is any sequence in ΠA, then there exists a $<_I$-increasing sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ of length λ in $\Pi A/I$, such that $g_\xi < f_{\xi+1}$ for every $\xi < \lambda$ and $(*)_\kappa$ holds for f for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A \mid a \leq \kappa^{++}\} \in I$. Hence if $\kappa = |A|^+$ is such a cardinal, then by Theorem 2.15 and the fact that $(*)_\kappa$ holds, we have an exact upper bound g to the sequence f so that $\{a \in A \mid \mathrm{cf}\, g(a) < \kappa\} \in I$.*

Proof. We shall define a $<_I$-increasing sequence $\langle f_\xi \mid \xi < \lambda\rangle$ in $\Pi A/I$ as follows. At successor stages, if f_ξ is defined, let $f_{\xi+1}$ be any function in ΠA that $<$-extends f_ξ and g_ξ.

1. At limit stages $\delta < \lambda$ there are two cases. In the first $\mathrm{cf}(\delta) = \kappa^{++}$ where κ is regular and $\{a \in A \mid a \leq \kappa^{++}\} \in I$. Then fix some $E_\delta \subseteq \delta$ closed unbounded and of order type $\mathrm{cf}(\delta)$, and define

$$f_\delta = \sup\{f_i \mid i \in E_\delta\}.$$

   Then $f_\delta(a) < a$ when $a > \kappa^{++}$, and thus $f_\delta \in \Pi A/I$ since $\{a \in A \mid a \leq \kappa^{++}\} \in I$.

2. If $\delta < \lambda$, but case 1 above does not hold, let $f_\delta \in \Pi A$ be any $\leq_I$ upper bound of $\langle f_\xi \mid \xi < \delta\rangle$ guaranteed by the λ-directedness assumption.

Now Lemma 2.19 implies that $(*)_\kappa$ holds for every regular cardinal κ of the required form. $\dashv$

In the following, we shall apply Lemma 2.19 (or rather its consequence Theorem 2.21 above) and obtain an important representation of successors of singular cardinals with uncountable cofinality. But first we introduce a notation.

2.22 Notation. Let X be a set of cardinals, then

$$X^{(+)} = \{\alpha^+ \mid \alpha \in X\}$$

denotes the set of successors of cardinals in X.

2.23 Theorem (Representation of μ^+ as true cofinality). *Suppose that μ is a singular cardinal with uncountable cofinality. Then there exists a closed unbounded set $C \subseteq \mu$ such that*

$$\mu^+ = \operatorname{tcf}(\Pi C^{(+)}/J^{bd})$$

where J^{bd} is the ideal of bounded subsets of $C^{(+)}$.

Proof. Let $C_0 \subseteq \mu$ be any closed unbounded set of limit cardinals such that $|C_0| = \operatorname{cf}(\mu)$ and all cardinals in C_0 are above $\operatorname{cf}(\mu)$. Clearly all cardinals in C_0 that are limit points of C_0 are singular cardinals, and hence we can assume that C_0 consists only of singular cardinals.

Observe that $\Pi C_0^{(+)}/J^{bd}$ is μ directed, and in fact is μ^+ directed since μ is a singular cardinal. Indeed, suppose that $F \subseteq \Pi C_0^{(+)}$ has cardinality $< \mu$ and define $h(a)$ by $h(a) = \sup\{f(a) \mid f \in F\}$ for every $a \in C_0^{(+)}$ above $|F|$ (so that $h(a) \in a$), and $h(a)$ is arbitrarily defined on smaller a's. This proves that every subset of $\Pi C_0^{(+)}$ of cardinality $< \mu$ is bounded in $<_{J^{bd}}$. But then it follows that subsets of $\Pi C_0^{(+)}$ of cardinality μ are also bounded: decompose any such subset $F = \bigcup_{\alpha < \operatorname{cf}(\mu)} F_\alpha$ where each F_α has cardinality $< \mu$, then bound each F_α, and finally bound the sequence of bounds.

Thus $\Pi C_0^{(+)}/J^{bd}$ is μ^+ directed and we may construct a J^{bd} increasing sequence $f = \langle f_\xi \mid \xi < \mu^+ \rangle$ in $\Pi C_0^{(+)}$ such that $(*)_\kappa$ holds for every regular cardinal $\kappa < \mu$ (apply Theorem 2.21 in its simpler form in which there is no need to extend a given sequence g).

Theorem 2.15 implies that f has an exact upper bound $h : C_0^{(+)} \to \mathrm{On}$ such that

$$\{a \in C_0^{(+)} \mid \operatorname{cf}(h(a)) < \kappa\} \in J^{bd} \tag{I.13}$$

for every regular $\kappa < \mu$. We may assume that $h(a) \leq a$ for every $a \in C_0^{(+)}$, since the identity function is clearly an upper bound to f.

2.24 Claim. *The set* $\{\alpha \in C_0 \mid h(\alpha^+) = \alpha^+\}$ *contains a closed unbounded set.*

Proof of Claim. Suppose toward a contradiction that for some stationary set $S \subseteq C_0$, $h(\alpha^+) < \alpha^+$ for every $\alpha \in S$. Since all cardinals of C_0 are singular, $\mathrm{cf}(h(\alpha^+)) < \alpha$ for every $\alpha \in S$. Hence (by Fodor's theorem) $\mathrm{cf}(h(\alpha^+))$ is bounded by some $\kappa < \mu$ on a stationary subset of α in S. But this is in contradiction to (I.13) above.

Thus we have proved the existence of a closed unbounded set $C \subseteq C_0$ such that $h(\alpha^+) = \alpha^+$ for every $\alpha \in C$. We claim that $\mu^+ = \mathrm{tcf}(\Pi C^{(+)}/J^{bd})$. But this is clear since $h \restriction C^{(+)}$, which is the identity function, is an exact upper bound to the sequence $\langle f_\xi \restriction C^{(+)} \mid \xi < \mu^+\rangle$ which is J^{bd} increasing and of length μ^+. This ends the proof of the claim and Theorem 2.23. $\dashv$

A somewhat stronger form of this theorem is in Exercise 4.17.

2.25 Exercise. Prove the following representation theorem for μ^+ in case $\mathrm{cf}(\mu) = \aleph_0$.

2.26 Theorem. *If* μ *is a singular cardinal of countable cofinality then for some unbounded set* $D \subseteq \mu$ *(of order type* ω*) of regular cardinals*

$$\mu^+ = \mathrm{tcf}(\Pi D/J^{bd})$$

where J^{bd} *is the ideal of bounded subsets of* D*. For example, there exists a set* $B \subseteq \{\aleph_n \mid n < \omega\}$ *such that* $\mathrm{tcf}\, \Pi B/J^{bd} = \aleph_{\omega+1}$.

Hint. Let C_0 be any ω sequence converging to μ, consisting of regular cardinals. Repeat the proof above and define $D = \{\mathrm{cf}(h(a)) \mid a \in C_0\}$. Then use Lemma 2.3.

The theory of exact upper bounds, which is the basis of the pcf theory, can be developed in various ways. For example [10] presents Shelah's Trichotomy theorem, and extends it to further analyze the set of flat points. Suitably interpreted, Theorem 2.15 is equivalent to Theorem 18 of [10]. The following exercise establishes the connection between the Trichotomy and the bounding projection property.

2.27 Exercise. (*The Trichotomy theorem.*) Suppose that $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi \mid \xi < \lambda\rangle$ is a $<_I$ increasing sequence. Consider the following properties of f and a regular cardinal κ such that $|A| < \kappa \leq \lambda$:

Bad$_\kappa$ There are sets of ordinals $S(a)$ for $a \in A$ such that $|S(a)| < \kappa$ and sup_of_S $<_I$-dominates f, and there is an ultrafilter D over A, extending the dual of I, such that for every $\alpha < \lambda$ $f^+_\alpha <_D f_\beta$ for some $\beta < \lambda$ (where $f^+_\alpha = \mathrm{proj}(f_\alpha, S)$).

Ugly There exists a function $g \in \mathrm{On}^A$ such that, forming $t_\alpha = \{a \in A \mid g(a) < f_\alpha(a)\}$, the sequence $\langle t_\alpha \mid \alpha < \lambda \rangle$ which we know to be $\subseteq_I$ increasing does not stabilize modulo I. That is, for every α there is some $\beta > \alpha$ in λ such that $t_\beta \setminus t_\alpha \in I^+$.

Good$_\kappa$ There exists an exact upper bound g to the sequence f such that $\mathrm{cf}(g(a)) \geq \kappa$ for every $a \in A$.

Prove that the bounding projection property for κ is equivalent to $\neg$**Bad**$_\kappa \wedge \neg$**Ugly**. Hence the Trichotomy theorem which says that if neither **Bad**$_\kappa$ nor **Ugly**, then **Good**$_\kappa$.

2.28 Exercise. (Lemma 0.D, Chapter V, [14]) If λ is a regular cardinal and $\forall \mu < \lambda\ \mu^{|A|} < \lambda$, if $f_\alpha \in \mathrm{On}^A$ for $\alpha < \lambda$, then for some unbounded $E \subseteq \lambda$, for all $\alpha < \beta$ both in E, $f_\alpha \leq f_\beta$ and $\{a \in A \mid f_\alpha(a) = f_\beta(a)\}$ does not depend on α, β in E.

Hence, if I is an ideal over A and $f_\alpha <_I f_\beta$ for all $\alpha < \beta$, then $(*)_\lambda$ holds.

Hint. For $a \in A$ fix some $\gamma_a > \sup\{f_\alpha(a) \mid \alpha \in \lambda\}$. For $\alpha < \lambda$, $a \in A$, define $s^\alpha(a) = \{f_\beta(a) \mid \beta < \alpha\} \cup \{\gamma_a\}$. Define $g_\alpha = \mathrm{proj}(f_\alpha, s^\alpha)$. Let $T = \{\delta < \lambda \mid \mathrm{cf}(\delta) = |A|^+\}$. For $\alpha \in T$ there exists $\mu_\alpha < \alpha$ such that $g_\alpha = \mathrm{proj}(f_\alpha, s^{\mu_\alpha})$. By Fodor's theorem we may assume $\mu = \mu_\alpha$ is fixed on a stationary set $T' \subseteq T$. Moreover, since s^μ has cardinality $|\mu|$ and $|\mu|^{|A|} < \lambda$, we may assume that $g_\alpha = g$ is fixed for $\alpha \in T'' \subseteq T'$, stationary.

2.2. Application: Silver's Theorem

One form of Silver's theorem says that if κ is a singular cardinal of uncountable cofinality such that $2^\delta = \delta^+$ for a stationary set of δ's in κ, then $2^\kappa = \kappa^+$. A slightly more general form is the following

2.29 Theorem. *(Silver [17]) Let κ be a singular cardinal with uncountable cofinality: $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$. Suppose that there exists a stationary set of cardinals $S \subseteq \kappa$ such that, for every $\delta \in S$, $\delta^{\mathrm{cf}(\kappa)} = \delta^+$. Then*

$$\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$$

as well.

Proof. Assume that $S \subseteq \kappa$, of order type $\mathrm{cf}(\kappa)$, is a stationary set of cardinals such that for every $\delta \in S$

$$\delta^{\mathrm{cf}(\kappa)} = \delta^+.$$

We have established the existence of a closed unbounded subset $C \subseteq \kappa$ with $\kappa^+ = \mathrm{tcf}(\Pi C^{(+)}/J^{bd})$. So, by taking $S \cap C$ for S, we may conclude that

$\Pi S^{(+)}/J_{bd}$ has true cofinality κ^+ and let $f = \langle f_\xi \mid \xi \in \kappa^+\rangle$ be J^{bd} increasing and cofinal there.

Since $\lambda^{\mathrm{cf}(\kappa)} = \lambda^+$ for all $\lambda \in S$, there exists an encoding of all pairs $\langle \lambda, X\rangle$ where $X \in [\lambda]^{\mathrm{cf}(\kappa)}$ by ordinals in λ^+. Hence we can encode each $X \in [\kappa]^{\mathrm{cf}(\kappa)}$ by a function $h_X \in \Pi S^{(+)}$, where $h_X(\lambda^+)$ gives the code of $X \cap \lambda$. Thus, if $X \neq Y$ then h_X and h_Y are eventually disjoint. Since each h_X is J^{bd} dominated by some f_ξ for $\xi \in \kappa^+$, the following lemma concludes the proof of our theorem.

2.30 Lemma. *For every function $g \in \Pi S^{(+)}$, the collection*

$$F = \{X \in [\kappa]^{\mathrm{cf}(\kappa)} \mid h_X <_{J^{bd}} g\}$$

has cardinality $\leq \kappa$.

Proof. Suppose that, on the contrary, $|F| \geq \kappa^+$. For each $\delta \in S$ fix an enumeration of $g(\delta^+) \in \delta^+$ that has order-type $\leq \delta$. Using this enumeration, $h_X(\delta^+)$ is "viewed" as an ordinal in δ, denoted $k_X(\delta)$ whenever $h_X(\delta^+) < g(\delta^+)$. Thus for every $X \in F$, h_X is translated into a pressing down function defined on a final segment of S.

By Fodor's theorem, for some stationary set $S_X \subseteq S$, k_X is bounded on S_X, say by $\delta_X < \kappa$. Now the number of subsets of S is bounded by $2^{\mathrm{cf}(\kappa)} < \kappa$, and hence there exists a subset $F_0 \subseteq F$ of cardinality κ^+, a fixed stationary set S_0, and a fixed cardinal $\delta_0 \in S$ such that $S_X = S_0$ and $\delta_X = \delta_0$ for every X in F_0. Moreover the translation function taking $\delta \in S_0$ to that ordinal in δ_0 that indirectly encodes $X \cap \delta$ can also be assumed to be independent of $X \in F_0$, since there are at most $\delta_0^{\mathrm{cf}(\kappa)} = \delta_0^+$ such functions. Yet this is absurd because the translation function of h_X completely determines $X = \bigcup\{X \cap \delta \mid \delta \in S_0\}$. A contradiction which proves the lemma and the theorem. $\dashv$

2.31 Exercise. Show that the following form of Silver's theorem is equivalent to 2.29 (cf. [8]). Let κ be a singular cardinal with uncountable cofinality: $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$. Suppose that $\lambda^{\mathrm{cf}(\kappa)} < \kappa$ for all $\lambda < \kappa$, and there exists a stationary set of cardinals $S \subseteq \kappa$ such that, for every $\delta \in S$, $\delta^{\mathrm{cf}(\delta)} = \delta^+$. Then

$$\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$$

as well.

2.32 Exercise.

The proof given by Baumgartner and Prikry (in [1]) to Silver's theorem simplifies the original proof, and is actually simpler than the proof given here which serves to illustrate some of the pcf concepts. In addition the Baumgartner–Prikry proof relies on very elementary notions. The following exercise describes that proof. Assume that κ is a singular cardinal with uncountable cofinality.

1. If $S \subset \kappa$ is a stationary set such that $\delta^{\mathrm{cf}(\kappa)} = \delta^+$ for $\delta \in S$, define on $\Pi S^{(+)}$ a relation R by $f\ R\ g$ iff $\{\alpha \in S \mid f(\alpha^+) < g(\alpha^+)\}$ is stationary. Prove that for every g the cardinality of $R^{-1}g = \{X \in \kappa^{\mathrm{cf}(\kappa)} \mid h_X\ R\ g\}$ is $\leq \kappa$.

2. Prove that for every $f, g \in \Pi S^{(+)}$ that are eventually different either $f\ R\ g$ or $g\ R\ f$. Take any collection $X_i \in [\kappa]^{\mathrm{cf}(\kappa)}$, $i < \kappa^+$, of different subsets and consider $H = \bigcup_{i \in \kappa^+} R^{-1}h_{X_i}$. If $\kappa^{\mathrm{cf}(\kappa)} > \kappa^+$ there must be some $g \notin H$, and hence $h_{X_i}\ R\ g$ for every $i < \kappa^+$. This is a contradiction.

2.3. Application: a covering theorem

In this subsection V denotes the universe of all sets, and U a transitive subclass containing all ordinals and satisfying the axioms of ZFC. (See, for example, Levy [12] for the meaning of statements concerning classes.) If X and Y are sets of ordinals in V and U (respectively) and $X \subseteq Y$, then we say that Y *covers* X.

The *countable covering property* of U (or between U and V) is the statement that any countable set of ordinals X is covered by some countable set of ordinals Y in U (that is, Y is in U, and Y is countable in V). Similarly, for any cardinal κ, the $\leq \kappa$ covering property is that any set of cardinals X of cardinality $\leq \kappa$ is covered by some set in U that has cardinality $\leq \kappa$ in V. If the $\leq \kappa$ covering property holds for every cardinal κ, then we say that the "full" covering property holds for U: every set of ordinals X is covered by some set Y in U such that X and Y are equinumerous in V. The following theorem gives conditions by which the full covering property can be deduced from the countable covering property.

2.33 Theorem. *(Magidor) Suppose U is a transitive class containing all ordinals and satisfying all ZFC axioms. Moreover, assume that*

1. *the GCH holds in U,*

2. *U and the universe V have the same cardinals, and moreover every regular cardinal in U remains regular in V.*

Then the countable covering property for U implies the full covering property.

Proof. Observe first that if U and V have the same regular cardinals, they have the same cardinals and $\mathrm{cf}^U(\kappa) = \mathrm{cf}^V(\kappa)$ for every ordinal. Also, for every set $X \in U$, $|X|^U = |X|^V$. We prove by induction on $\lambda \in \mathrm{On}$ that every $X \subseteq \lambda$ is covered by some Y in U of the same V cardinality. Of course if X is bounded in λ then the inductive assumption applies, and hence we can consider only sets that are unbounded in λ.

If λ is not a cardinal, let $|\lambda|$ be its cardinality. So $|\lambda| < \lambda < \lambda^+$ in U as well since V and U have the same cardinals. Since λ and $|\lambda|$ are equinumerous in U, the inductive assumption for $|\lambda|$ implies that any subset of λ can be covered by a set in U of the same cardinality.

So we assume that λ is a cardinal. If it is a regular cardinal, then any unbounded $X \subseteq \lambda$ is covered by λ itself. Hence we are left with the case that λ is a singular cardinal, in V and hence in U since both universes have the same regular/singular predicate. Again, if $X \subseteq \lambda$ has cardinality λ then λ itself is a covering as required, and hence we may assume that $|X| < \lambda$.

Assume first that $\mathrm{cf}(\lambda) = \omega$. Then $\mathrm{cf}^U(\lambda) = \omega$ as well. Suppose that an unbounded set $X \subseteq \lambda$ of cardinality $< \lambda$ is given. Take in U an increasing cofinal in λ sequence $\langle \lambda_i \mid i \in \omega \rangle$. Since $2^{<\lambda} = \lambda$ is assumed in U, there exists in U an enumeration of length λ of all *bounded* subsets of λ of cardinality $\leq |X|$. Now consider the sequence $X \cap \lambda_i$, $i \in \omega$ (where X is the set to be covered) and cover first each $X \cap \lambda_i$ by some $Y_i \in U$ with $|Y_i| = |X \cap \lambda_i|$. Then define $\alpha_i \in \lambda$ to be the ordinal that encodes Y_i in U, and form the countable set $A = \{\alpha_i \mid i \in \omega\}$ of ordinals that encode X. This countable set A can be covered by some countable set A' in U, and we can define in U a cover

$$Y = \bigcup \{E \subseteq \lambda_i \mid \ E \text{ is encoded by some ordinal in } A' \text{ and } |E| \leq |X|\}.$$

Clearly $X \subseteq Y$ and $|Y| = |X|$.

Finally suppose that λ is a singular cardinal with uncountable cofinality, and it is here that the theory developed so far is employed. Since V and U have the same regular cardinals $\mathrm{cf}^U(\lambda) = \mathrm{cf}(\lambda)$.

We work for a while in U and apply the Representation Theorem 2.23 to λ. In fact, we must analyze the proof and use the construction rather than the theorem. Recall that we took an arbitrary closed unbounded set $C_0 \subseteq \lambda$ consisting of singular cardinals, and such that $|C_0| = \mathrm{cf}(\lambda) < \min\ C_0$. Then we constructed a J^{bd} increasing sequence $f = \langle f_\xi \mid \xi < \lambda^+ \rangle$ in $\Pi C_0^{(+)}$ such that for all limit ordinals $\delta < \lambda^+$ a closed unbounded set $E_\delta \subseteq \delta$ was chosen with $|E_\delta| = \mathrm{cf}(\delta) < \lambda$ and then

$$f_\delta =_{J^{bd}} \sup\{f_i \mid i \in E_\delta\}$$

was defined. All of this is done in U, but now we pass to V and deduce that $(*)_\kappa$ holds for every regular $\kappa < \lambda$ (by Lemma 2.19). Hence f has an exact upper bound h such that $\{a \in C_0^{(+)} \mid \mathrm{cf}(h(a)) < \kappa\} \in J^{bd}$ for every $\kappa < \lambda$. Now the argument of Claim 2.24 applies, and there exists (in V) a closed unbounded set $C \subseteq C_0$ such that $\{f_\xi \restriction C^{(+)} \mid \xi < \lambda^+\}$ is cofinal in $\Pi C^{(+)}/J^{bd}$.

We continue now the proof that any set $X \subseteq \lambda$ of cardinality $\lambda_0 < \lambda$ can be covered in U by a set of the same cardinality. Since X is unbounded in

λ, $\lambda_0 \geq \mathrm{cf}(\lambda)$. For every $\alpha \in C_0$ cover $X \cap \alpha$ by some $Y_\alpha \in U$ (a subset of α) of cardinality $\leq \lambda_0$. We assume in U an enumeration of length α^+ of all subsets of α of size $\leq \lambda_0$. There is an index $< \alpha^+$ that encodes Y_α in U. The function d taking $\alpha^+ \in C_0^{(+)}$ to that coding ordinal is defined in V and is bounded by some $f_\xi \in U$. Namely $d \upharpoonright C^{(+)} <_{J^{bd}} f_\xi \upharpoonright C^{(+)}$. In U, choose for every $\alpha \in C_0$ a function $g_\alpha : f_\xi(\alpha^+) \to \alpha$ that is one-to-one. Then in V look at the values $g_\alpha(d(\alpha^+)) < \alpha$, and find a stationary set $S \subseteq C$ on which these values are bounded, say by κ. The set $\{g_\alpha(d(\alpha)) \mid \alpha \in S\} \subseteq \kappa$ can be covered by some set Y in U that has the same cardinality (namely $\mathrm{cf}(\lambda)$). Now look in U at the set $\bigcup_{\alpha \in C_0} g_\alpha^{-1} Y$. Every index in $g_\alpha^{-1} Y$ represents a subset of α of cardinality $\leq \lambda_0$, and hence this yields a cover of X of cardinality λ_0. $\dashv$

2.34 Exercise. There is actually no need to start with countable covering in order to deduce covering for all higher cardinals. The following generalization is left as an exercise.

2.35 Theorem. *As in Theorem 2.33 assume that V and $U \subseteq V$ have the same regular cardinals, and the GCH holds in U. Let λ_0 be any cardinal such that every countable set of ordinals is covered by some set in U of cardinality $\leq \lambda_0$. Then any set of ordinals X is covered by some set in U of cardinality $|X| + \lambda_0$.*

3. Basic properties of the pcf function

For any set A of regular uncountable cardinals define

$$\mathrm{pcf}(A) = \{\lambda \mid \text{for some ultrafilter } U \text{ over } A,\ \lambda = \mathrm{cf}(\Pi A/U)\}.$$

Some easily verifiable properties:

1. If $\lambda = \mathrm{tcf}(\Pi A/F)$ for some filter F over A, then $\lambda \in \mathrm{pcf}(A)$. (For any ultrafilter U that extends F $\lambda = \mathrm{tcf}(\Pi A/U)$.)

2. $A \subseteq \mathrm{pcf}(A)$. Since for every $a \in A$ we can take the principal ultrafilter over A concentrating on $\{a\}$.

3. $A \subseteq B$ implies $\mathrm{pcf}(A) \subseteq \mathrm{pcf}(B)$. Because every ultrafilter D over A can be extended to D' over B, and the ultraproducts $\Pi A/D$ and $\Pi B/D'$ are the same.

4. For any sets A and B, $\mathrm{pcf}(A \cup B) = \mathrm{pcf}(A) \cup \mathrm{pcf}(B)$. Indeed, if $\lambda \in \mathrm{pcf}(A \cup B)$, and D is an ultrafilter over $A \cup B$ with ultraproduct of cofinality λ, then either $A \in D$ or $B \in D$ (or both) and hence $\lambda \in \mathrm{pcf}(A)$ or $\lambda \in \mathrm{pcf}(B)$. For the other direction use the previous item.

We say that A is an *interval of regular cardinals* if for some cardinals $\alpha < \beta$, A is the set of all regular cardinals κ such that $\alpha \le \kappa < \beta$. This term is slightly misleading because one may misinterpret it as saying that all cardinals between α and β are regular.

3.1 Theorem (The "no holes" argument). *Assume that A is an interval of regular cardinals satisfying $|A| < \min A$, and λ is a regular cardinal with $\sup A < \lambda$. Let I be a proper ideal over A such that $\Pi A/I$ is λ directed. Then $\lambda \in \mathrm{pcf}(A)$.*

Proof. We may assume that every proper initial segment of A is in I (or else substitute for A its first initial segment that is not in I.) It now follows that A is infinite and unbounded (without a maximum).

Theorem 2.21 gives an $<_I$-increasing sequence $f = \langle f_\xi \mid \xi \in \lambda \rangle$ in $\Pi A/I$ that satisfies $(*)_\kappa$ for every regular cardinal κ in A (and thus for smaller cardinals of course). In particular $(*)_{|A|^+}$ holds, and f has an exact upper bound $h \in \mathrm{On}^A$ such that

$$\{a \in A \mid \mathrm{cf}(h(a)) < \kappa\} \in I \tag{I.14}$$

for every $\kappa \in A$ (this by Theorem 2.15). Since the identity function $\mathrm{id} : A \to A$ taking a to a is clearly an upper bound of f, $h(a) \le a$ for I-almost all $a \in A$. Yet (I.14) implies that

$$\{a \in A \mid \mathrm{cf}(h(a)) < \min A\} \in I,$$

and hence we have $\min(A) \le \mathrm{cf}(h(a)) \le a$ for I-almost all $a \in A$. Changing h on a null set, we may assume for simplicity that this holds for every $a \in A$, namely that

$$\mathrm{cf}(h(a)) \in A \quad \text{for all } a \in A$$

(as A is an interval of regular cardinals). Since the sequence f has length λ, $\Pi h/I$ has true cofinality λ. Consequently $\Pi_{a\in A}\mathrm{cf}(h(a))/I$ has true cofinality λ as well. Since $|A| < \min A$, Lemma 2.3 gives a proper ideal J on $B = \{\mathrm{cf}(h(a)) \mid a \in A\} \subseteq A$, such that $\Pi B/J$ has true cofinality λ as well. So $\lambda \in \mathrm{pcf}(A)$. We note in addition that J is the Rudin–Keisler projection obtained via $\mathrm{cf} \circ h$, and hence (I.14) implies for every $\kappa < \sup(A)$ that $B \cap \kappa \in J$. $\dashv$

Upon examination of the proof, the reader will notice that the following slightly stronger formulation of the theorem can be obtained. In this formulation the requirement that A is an interval is relaxed.

3.2 Theorem. *Assume that A is a set of regular cardinals such that $|A| < \min A$, and λ is a regular cardinal such that $\sup A < \lambda$. Suppose that I is a proper ideal over A containing all proper initial segments of A and such that $\Pi A/I$ is λ-directed. Then $\lambda \in \mathrm{pcf}(A')$ for some set A' of regular cardinals such that*

1. *$A' \subseteq [\min(A), \sup(A))$, and A' is cofinal in* $\sup(A)$.

2. $|A'| \leq |A|$.

In fact, λ is the true cofinality of $\Pi A'/J$ for an ideal J over A' that contains all bounded subsets of A'.

Proof. Follow the previous proof and let A' be the set $\{\mathrm{cf}(h(a)) \mid a \in A\}$. $\dashv$

3.3 Notation. The property $|A| < \min A$ assumed for the set of regular cardinals appearing in the theorem is so pervasive in the pcf theory that it ought to be given a name. Following [6] we say that a set of regular cardinals A is *progressive* if $|A| < \min A$.

3.1. The ideal $J_{<\lambda}$

Let A be a set of regular cardinals. For any cardinal λ define

$$J_{<\lambda}[A] = \{X \subseteq A \mid \mathrm{pcf}(X) \subseteq \lambda\}.$$

In plain words, $X \in J_{<\lambda}[A]$ iff for every ultrafilter D over A such that $X \in D$, $\mathrm{cf}(\Pi A/D) < \lambda$. That is, X "forces" the cofinalities of its ultraproducts to be below λ.

Clearly $J_{<\lambda}[A]$ is an ideal over A, but it is not necessarily a proper ideal since $A \in J_{<\lambda}[A]$ is possible. However, if $\lambda \in \mathrm{pcf}(A)$, then $J_{<\lambda}[A]$ *is* proper ($A \notin J_{<\lambda}[A]$, or else $\mathrm{pcf}(A) \subseteq \lambda$ shows that $\lambda \notin \mathrm{pcf}(A)$). When the identity of A is obvious from the context, we write $J_{<\lambda}$ instead of $J_{<\lambda}[A]$. Note that if $A \subseteq B$ then $J_{<\lambda}[A] = J_{<\lambda}[B] \cap \mathcal{P}(A)$.

Let $J^*_{<\lambda}[A]$ be the dual filter over A. Then

$$J^*_{<\lambda}[A] = \bigcap\{D \mid D \text{ is an ultrafilter and } \mathrm{cf}(\Pi A/D) \geq \lambda\}.$$

3.4 Theorem (λ-Directedness). *Assume that A is a progressive set of regular cardinals. Then, for every cardinal λ, $\Pi A/J_{<\lambda}[A]$ is λ-directed: any set of less than λ functions is bounded in $\Pi A/J_{<\lambda}[A]$.*

Proof. The theorem holds trivially if $A \in J_{<\lambda}[A]$, since $|\Pi A/J_{<\lambda}| = 1$ in this case. So we assume that $J_{<\lambda}$ is a proper ideal over A. Let $\kappa_0 = \min A$ be the first cardinal of A, and κ_1, κ_2 be the second, third etc. cardinals of A. The case $\lambda \leq \kappa_n$ for n finite is quite obvious: if $\lambda = \kappa_n$ then $J_{<\lambda} = \mathcal{P}(\{\kappa_0, \dots, \kappa_{n-1}\})$ and for every family $F \subseteq \Pi A$ of cardinality $< \lambda$, $\sup F \in \Pi A$, because $(\sup F)(a) = \bigcup\{f(a) \mid f \in F\} < a$, since $|F| < \lambda \leq a$ for every $a \notin \{\kappa_0, \dots, \kappa_{n-1}\}$). So we can certainly assume that $\lambda > \kappa_n$ for all $n \in \omega$, and hence that $\{\kappa_n\} \in J_{<\lambda}$.

Since any null subset of A can be removed without changing the structure of $\Pi A/J_{<\lambda}$, we may assume that $|A|^+, |A|^{++}, |A|^{+3} \notin A$. That is we can assume that

$$|A|^{+3} < \min A < \lambda.$$

We shall prove by induction on $\lambda_0 < \lambda$ that $\Pi A/J_{<\lambda}$ is λ_0^+-directed: for every $F = \{f_i \mid i \in \lambda_0\} \subseteq \Pi A$ a family of functions of cardinality λ_0, F has an upper bound in $\Pi A/J_{<\lambda}$. The case $\lambda_0 < \min(A)$ is obvious as we saw.

So let $F = \{f_i \mid i \in \lambda_0\} \subseteq \Pi A$ be a subset of ΠA where $\lambda_0 < \lambda$ and assume that $\Pi A/J_{<\lambda}$ is λ_0-directed. Our aim is to bound F in $\Pi A/J_{<\lambda}$.

In case λ_0 is singular, we take $\langle \alpha_i \mid i < \mathrm{cf}(\lambda_0)\rangle$ increasing and cofinal in λ_0, and obtain $g_i \in \Pi A$ for every $i < \mathrm{cf}(\lambda_0)$ that bounds $\{f_\xi \mid \xi < \alpha_i\}$. Then we apply the inductive assumption again to the sequence $\{g_i \mid i < \mathrm{cf}(\lambda_0)\}$, and obtain a bound to F.

Thus λ_0 is assumed to be a regular cardinal above $|A|^{+3}$. We shall replace F by a $J_{<\lambda}$ increasing sequence that satisfies $(*)_\kappa$ for $\kappa = |A|^+$. That is, using Theorem 2.21 we define a $J_{<\lambda}$-increasing sequence $\langle f'_\xi \mid \xi < \lambda_0\rangle$ satisfying $(*)_\kappa$ and such that $f_i \leq f'_i$.

Hence we can assume that the sequence $f = \langle f_i \mid i < \lambda_0\rangle$ that we want to dominate satisfies $(*)_{|A|^+}$ and thus has an exact upper bound $g \in \mathrm{On}^A$ in $<_{J_{<\lambda}[A]}$ (by Theorem 2.15).

Since the identity function taking $a \in A$ to a is an upper bound of our sequence f, we may assume that $g(a) \leq a$ for all $a \in A$ (by possibly changing g on a null set). We intend to prove that $B = \{a \in A \mid g(a) = a\} \in J_{<\lambda}[A]$, and thus that $g =_{J_{<\lambda}} g'$ for some $g' \in \Pi A$ which will show that g bounds f in $\Pi A/J_{<\lambda}[A]$.

Assume toward a contradiction that $B \notin J_{<\lambda}[A]$. Then (by definition of $J_{<\lambda}$) there is an ultrafilter D over A such that $B \in D$ and $\mathrm{cf}(\Pi A/D) \geq \lambda$. Clearly $D \cap J_{<\lambda} = \emptyset$, or else $\mathrm{cf}(\Pi A/D) < \lambda$. The sequence f of length $\lambda_0 < \lambda$ is necessarily bounded in $\Pi A/D$ and we let $h \in \Pi A/D$ be such a bound. So $h(a) < g(a)$ for every $a \in B$ (since $g(a) = a$ for $a \in B$). Hence (by definition of an exact upper bound) there is some f_i in f such that $h \restriction B <_{J_{<\lambda}[A]} f_i \restriction B$. But this would imply $h <_D f_i$, which contradicts the definition of h as an upper bound. $\dashv$

3.5 Corollary. *Suppose that A is a progressive set of regular cardinals. For every ultrafilter D over A*

$$\mathrm{cf}(\Pi A/D) < \lambda \textit{ iff } J_{<\lambda}[A] \cap D \neq \emptyset.$$

Hence $\mathrm{cf}(\Pi A/D) = \lambda$ *iff* $J_{<\lambda^+} \cap D \neq \emptyset$ *and* $J_{<\lambda} \cap D = \emptyset$. *Namely,* $\mathrm{cf}(\Pi A/D) = \lambda$ *iff* λ^+ *is the first cardinal* μ *such that* $J_{<\mu} \cap D \neq \emptyset$.

Proof. If $J_{<\lambda}[A] \cap D \neq \emptyset$ and $X \in J_{<\lambda}[A] \cap D$, then by definition of $X \in J_{<\lambda}$

$$\mathrm{cf}(\Pi A/D) < \lambda.$$

On the other hand, if $J_{<\lambda} \cap D = \emptyset$, then the above theorem stating that $\Pi A/J_{<\lambda}$ is λ-directed gives that $\Pi A/D$ is λ-directed as well. Thus $\mathrm{cf}(\Pi A/D) < \lambda$ is impossible in this case. The additional conclusion of the corollary is easily derived. $\dashv$

This corollary allows us to investigate the relationship between $J_{<\lambda}[A]$ and $J_{<\lambda^+}[A]$. By definition $X \in J_{<\lambda^+}[A]$ iff $X \subseteq A$ and for every ultrafilter D over A containing X, $\mathrm{cf}(\Pi A/D) \leq \lambda$. For this reason, $J_{<\lambda^+}[A]$ is also denoted $J_{\leq\lambda}[A]$.

If $\lambda \notin \mathrm{pcf}(A)$, for example when λ is singular, then $J_{<\lambda} = J_{\leq\lambda}$. However, if $\lambda \in \mathrm{pcf}(A)$ then $J_{<\lambda} \subset J_{\leq\lambda}$ (where $\subset$ is the strict inclusion relation). Indeed, if D is an ultrafilter over A such that $\mathrm{cf}(\Pi A/D) = \lambda$, then by Corollary 3.5 applied to λ^+, $J_{\leq\lambda} \cap D \neq \emptyset$, and certainly $J_{<\lambda} \cap D = \emptyset$. This argument shows that there is a one-to-one mapping from $\mathrm{pcf}(A)$ into $\mathcal{P}(A)$. Namely choosing $X_\lambda \in J_{\leq\lambda} \setminus J_{<\lambda}$ for every $\lambda \in \mathrm{pcf}(A)$. Thus we have the following theorem which is not evident from the definition of pcf.

3.6 Theorem. *If A is a progressive set of regular cardinals, then*

$$|\mathrm{pcf}(A)| \leq |\mathcal{P}(A)|$$

Another consequence of Theorem 3.4 is that $\max \mathrm{pcf}\ A$ exists.

3.7 Corollary (Max Pcf). *If A is a progressive set of regular cardinals, then the set* $\mathrm{pcf}(A)$ *contains a maximal cardinal.*

Proof. Observe that if $\lambda_1 < \lambda_2$ are cardinals, then $J_{<\lambda_1}[A] \subseteq J_{<\lambda_2}[A]$. Define

$$I = \bigcup \{J_{<\lambda}[A] \mid \lambda \in \mathrm{pcf}(A)\}.$$

For every $\lambda \in \mathrm{pcf}(A)$ $J_{<\lambda}[A]$ is a proper ideal on A, and hence I (being the union of a chain of proper ideals) is also a proper ideal.

Since I is a proper ideal it can be extended to a maximal proper ideal, and we let E be any ultrafilter over A and disjoint to I. Let $\mu = \mathrm{cf}(\Pi A/E)$. Since E is disjoint to I, it is disjoint to every $J_{<\lambda}[A]$ for $\lambda \in \mathrm{pcf}(A)$, and hence $\mathrm{cf}(\Pi A/E) \geq \lambda$ by the previous corollary. That is $\mu = \mathrm{cf}(\Pi A/E) = \max(\mathrm{pcf}\ A)$. As an important consequence we note that $\mu = \sup \mathrm{pcf}(A) = \max \mathrm{pcf}(A)$ is a regular cardinal (since it is in pcf A). $\dashv$

3.8 Exercise. If λ is a limit cardinal then

$$J_{<\lambda}[A] = \bigcup_{\theta<\lambda} J_{<\theta}[A].$$

Another way of writing this statement is that for every cardinal λ (not necessarily limit)

$$J_{<\lambda}[A] = \bigcup_{\theta<\lambda} J_{<\theta^+}[A] = \bigcup_{\theta<\lambda} J_{\leq\theta}[A].$$

The no holes argument has the following consequence.

3.9 Theorem. *Suppose that A is a progressive interval of regular cardinals. Then* $\mathrm{pcf}(A)$ *is again an interval of regular cardinals.*

Proof. We may assume that A is infinite, as the finite case is clear. We may also assume that A has no last cardinal (and deduce the general theorem in a short argument). Let $\lambda_0 = \max\ \mathrm{pcf}(A)$. We must show that every regular cardinal in the interval $[\min(A), \lambda_0]$ is in $\mathrm{pcf}(A)$. Say $\mu = \sup\ A$. Since $\mu \notin A$ (A has no maximum), μ is a singular cardinal (because A is progressive). Since $A \subseteq \mathrm{pcf}(A)$ and A is an interval of regular cardinals, the substantial part of the proof is in showing that any regular cardinal in $(\mu, \lambda_0]$ is in $\mathrm{pcf}(A)$. But if λ is a regular cardinal and $\mu < \lambda \leq \lambda_0$, then $J_{<\lambda}$ is a proper ideal (since $\lambda \leq \max\ \mathrm{pcf}(A)$). By Theorem 3.4, $\Pi A/J_{<\lambda}$ is λ-directed. Hence Theorem 3.1 applies, and $\lambda \in \mathrm{pcf}(A)$. $\dashv$

We can get some information even when A is not progressive.

3.10 Definition. Suppose that A is a set of regular cardinals and $\kappa < \min(A)$ is a cardinal. We define

$$\mathrm{pcf}_\kappa(A) = \bigcup\{\mathrm{pcf}(X) \mid X \subseteq A \text{ and } |X| = \kappa\}.$$

That is, $\mathrm{pcf}_\kappa(A)$ is the collection of all cofinalities of ultraproducts of A over ultrafilters that concentrate on subsets of A of power κ (or less).

Similarly to the previous theorem stating that $\mathrm{pcf}(A)$ of a progressive interval A is again an interval of regular cardinals, we have the following.

3.11 Theorem. *If A is an interval of regular cardinals, and $\kappa < \min\ A$, then* $\mathrm{pcf}_\kappa(A)$ *is an interval of regular cardinals.*

Proof. Define $\lambda_0 = \sup\ \mathrm{pcf}_\kappa(A)$, and let λ be a regular cardinal such that $\min(A) < \lambda < \lambda_0$. Then for some $X \subseteq A$ such that $|X| = \kappa$, $\lambda \leq \max\ \mathrm{pcf}(X)$. Hence $J_{<\lambda}[X]$ is proper, and we may assume that every initial segment of X is in $J_{<\lambda}$. As X *is* progressive, $\Pi X/J_{<\lambda}$ is λ-directed, Theorem 3.2 can be applied, and it yields that $\lambda \in \mathrm{pcf}(X')$ for some $X' \subseteq A$ of cardinality $\leq |X|$. Thus $\lambda \in \mathrm{pcf}_\kappa(A)$. $\dashv$

Yet another consequence of the λ-directedness of $\Pi A/J_{<\lambda}$ is the following

3.12 Theorem. *Suppose that A is a progressive set of regular cardinals and* $B \subseteq \mathrm{pcf}(A)$ *is also progressive, then*

$$\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A).$$

Hence if $\mathrm{pcf}(A)$ *is progressive, then* $\mathrm{pcf}\ \mathrm{pcf}(A) = \mathrm{pcf}(A)$.

Proof. Suppose that $\mu \in \mathrm{pcf}(B)$, and let E be an ultrafilter over B such that

$$\mu = \mathrm{cf}(\Pi_{b \in B}\ b/E). \tag{I.15}$$

For every $b \in B$ fix an ultrafilter D_b over A such that

$$b = \mathrm{cf}(\Pi A/D_b).$$

Define an ultrafilter D over A by

$$X \in D \text{ iff } \{b \in B \mid X \in D_b\} \in E. \tag{I.16}$$

We shall prove that $\mu = \mathrm{cf}(\Pi A/D)$, and hence that $\mu \in \mathrm{pcf}(A)$.

Consider (I.15). If, for every $b \in B$, $(b', <_{b'})$ is an ordering that has true cofinality b, then $\mu = \mathrm{cf}(\Pi_{b \in B}\ b'/E)$ as well. Hence

$$\mu = \mathrm{cf}(\Pi_{b \in B}(\Pi A/D_b)/E). \tag{I.17}$$

It remains to implement this iterated ultraproduct as an ultraproduct of A over D. For this aim consider the Cartesian product $B \times A$ and the ultrafilter P defined on $B \times A$ by

$$H \in P \text{ iff } \{b \in B \mid \{a \in A \mid \langle b, a\rangle \in H\} \in D_b\} \in E.$$

For any pair $\langle b, a\rangle$ let $r(\langle b, a\rangle) = a$ be its right projection. The reader should prove the following isomorphism

3.13 Claim. $\Pi_{\langle b,a\rangle \in B \times A}\ r(\langle b, a\rangle)/P \cong \Pi_{b \in B}(\Pi A/D_b)/E.$

Thus μ (an arbitrary cardinal in $\mathrm{pcf}(B)$) is the cofinality of the ultraproduct $\Pi_{\langle b,a\rangle \in B \times A}\ r(\langle b, a\rangle)/P$. But the projection map $r : B \times A \to A$, shows that the ultrafilter D defined in (I.16) is the Rudin–Keisler projection of P, and we are almost in the situation of Lemma 2.3, which concludes that $\mu = \mathrm{cf}(\Pi A/D)$. However Lemma 2.3 cannot be used verbatim because $|B \times A| < \min\ A$ is not assumed. All we know is that $|B| < \min\ B$. Recall (Lemma 2.3) that we had a map from ΠA into $\Pi_{\langle b,a\rangle \in B \times A}\ r(\langle b, a\rangle)$ carrying $h \in \Pi A$ to $\bar{h} \in \Pi_{\langle b,a\rangle \in B \times A}\ r(\langle b, a\rangle)$ defined by

$$\bar{h}(\langle b, a\rangle) = h(a).$$

We have proved that this map induces an isomorphism denoted L of $\Pi A/D$ into $\Pi_{\langle b,a\rangle \in B \times A}\ r(\langle b, a\rangle)$, but the problem is to prove that the image of L is cofinal there. Let $\lambda = \min\ B$. We have assumed that $|B| < \lambda$, and we shall use the fact that the reduced product modulo $J_{<\lambda}[A]$ is λ-directed as follows. Given any $g \in \Pi_{\langle b,a\rangle \in B \times A}\ r(\langle b, a\rangle)$ define for every $b \in B$ the map $g_b \in \Pi A$ by

$$g_b(a) = g(b, a).$$

Then $\{g_b \mid b \in B\}$ is bounded in $\Pi A/J_{<\lambda}[A]$ by some function $h \in \Pi A$, and we thus have that $g_b <_{J_{<\lambda}[A]} h$ for every $b \in B$. Hence

$$g_b <_{D_b} h$$

since $J_{<\lambda} \cap D_b = \emptyset$ (because $\mathrm{cf}(\Pi A/D_b) = b$ and $\lambda \leq b$). So $g <_P \bar{h}$ is concluded. $\dashv$

4. Generators for $J_{<\lambda}$

A very useful property of the $J_{<\lambda}$ ideals is that for every cardinal $\lambda \in \mathrm{pcf}(A)$ there is a set $B_\lambda \subseteq A$ such that

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda$$

which means that the ideal $J_{<\lambda^+}[A]$ is generated by the sets in $J_{<\lambda}[A] \cup \{B_\lambda\}$. That is, for every $X \subseteq A$, $X \in J_{<\lambda^+}$ iff $X \setminus B_\lambda \in J_{<\lambda}$. So B_λ is a maximal set in $J_{\leq\lambda}[A]$ in the sense that if $B_\lambda \subseteq C \in J_{\leq\lambda}$ then $C \setminus B_\lambda \in J_{<\lambda}$. The property that $J_{\leq\lambda}[A]$ is generated from $J_{<\lambda}[A]$ by the addition of a single set is called *normality*.

Normality of $\lambda \in \mathrm{pcf}(A)$ is obtained by means of a universal sequence for λ, and these sequences are studied first.

4.1 Definition. Suppose that $\lambda \in \mathrm{pcf}(A)$. A sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ of functions in ΠA, increasing in $<_{J_{<\lambda}}$, is a *universal* sequence for λ if and only if, for every ultrafilter D over A such that $\lambda = \mathrm{cf}(\Pi A/D)$, f is cofinal in $\Pi A/D$.

4.2 Theorem (Universally Cofinal Sequences). *Suppose that A is a progressive set of regular cardinals. Then every $\lambda \in \mathrm{pcf}(A)$ has a universal sequence.*

Proof. The proof is obvious in case $\lambda = \min\ A$. (The functions f_ξ defined by $f_\xi(a) = \xi$ will do.) Therefore we shall assume that $|A|^+ < \min\ A < \lambda$.

Suppose that there is no universal sequence for λ. This means that for every $J_{<\lambda}$-increasing sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ there is an ultrafilter D over A such that $\mathrm{cf}(\Pi A/D) = \lambda$ but f is bounded in $\Pi A/D$.

The proof is typical in that it makes $|A|^+$ steps and obtains a contradiction from the continuous failure at every step.

So for each $\alpha < |A|^+$ we shall define a $J_{<\lambda}$ increasing sequence $f^\alpha = \langle f^\alpha_\xi \mid \xi < \lambda\rangle$ in ΠA, and assume that no f^α is universal. The definition is by recursion on $\alpha < |A|^+$ and the fact that $\Pi A/J_{<\lambda}$ is λ directed is used in this construction.

If we visualize the functions f^α_ξ as lying on a matrix $\langle \xi, \alpha\rangle \in \lambda \times |A|^+$, then in each column α the functions f^α_ξ are $<_{J_{<\lambda}}$ increasing with ξ, and in each row ξ the functions f^α_ξ are $\leq$ increasing with α.

To begin with $f^0 = \langle f^0_\xi \mid \xi < \lambda \rangle$ is an arbitrary $<_{J_{<\lambda}}$-increasing sequence in $\Pi A/J_{<\lambda}$ of length λ.

At limit stages $\delta < |A|^+$ we define $f^\delta = \langle f^\delta_\xi \mid \xi < \lambda \rangle$ by induction on $\xi < \lambda$ so that for every $\xi < \lambda$

1. $f^\delta_i <_{J_{<\lambda}} f^\delta_\xi$ for $i < \xi$.
2. $\sup\{f^\alpha_\xi \mid \alpha < \delta\} \leq f^\delta_\xi$.

Suppose now that f^α is defined. Since it is not universal, there exists an ultrafilter D_α over A such that

1. $\mathrm{cf}(\Pi A/D_\alpha) = \lambda$, and
2. the sequence f^α is bounded in $<_{D_\alpha}$.

So we can choose $f^{\alpha+1}_0$ that bounds the sequence f^α in $<_{D_\alpha}$. The sequence $f^{\alpha+1}_i$ for $0 < i < \lambda$ is defined recursively by requiring that

1. $f^{\alpha+1}$ is $J_{<\lambda}$ increasing and cofinal in $\Pi A/D_\alpha$, and
2. $f^{\alpha+1}_i \geq f^\alpha_i$ (everywhere) for every $i < \lambda$.

To sum-up, we have constructed $<_{J_{<\lambda}}$-increasing sequences f^α, each of length λ, and ultrafilters D_α over A, for $\alpha < |A|^+$ so that:

1. for every $i < \lambda$, $\langle f^\alpha_i \mid \alpha < |A|^+ \rangle$ is increasing in $\leq$ (i.e., for $\alpha_1 < \alpha_2 < |A|^+$, $f^{\alpha_1}_i(a) \leq f^{\alpha_2}_i(a)$ for every a).
2. $f^\alpha = \langle f^\alpha_\xi \mid \xi < \lambda \rangle$ is bounded in $\Pi A/D_\alpha$ by $f^{\alpha+1}_0$.
3. $f^{\alpha+1}$ is cofinal in $\Pi A/D_\alpha$.

Now let $h = \sup\{f^\alpha_0 \mid \alpha < |A|^+\}$. Then $h \in \Pi A$, because $|A|^+ < \min\ A$. Find for every $\alpha < |A|^+$ an index $i_\alpha < \lambda$ such that $h <_{D_\alpha} f^{\alpha+1}_{i_\alpha}$. This is possible since $f^{\alpha+1}$ is cofinal in $\Pi A/D_\alpha$. Now pick an ordinal $i < \lambda$ such that $i > i_\alpha$ for every $\alpha < |A|^+$. This is possible since $\lambda > |A|^+$ is regular. So $h <_{D_\alpha} f^{\alpha+1}_i$ for every $\alpha < |A|^+$.

Define

$$A^\alpha \ = \leq(h, f^\alpha_i).$$

The sets $A^\alpha \subseteq A$ are increasing with α, that is $A^\alpha \subseteq A^\beta$ for $\alpha < \beta < |A|^+$ (since $f^\alpha_i \leq f^\beta_i$).

The contradiction is obtained when we show that $A^\alpha \subset A^{\alpha+1}$ (strict inclusion) for every $\alpha < |A|^+$ (and contrast this with $A^\alpha \subseteq A$). For this, observe the following two statements.

1. $A^\alpha \notin D_\alpha$, because $f^\alpha_i <_{D_\alpha} f^{\alpha+1}_0 \leq h$.
2. $A^{\alpha+1} \in D_\alpha$, because $h <_{D_\alpha} f^{\alpha+1}_i$.

$\dashv$

If $\lambda \in \mathrm{pcf}(A)$ and D is an ultrafilter over A such that $\mathrm{cf}(\Pi A/D) = \lambda$, then $A \cap (\lambda+1) \in D$ because otherwise $\{a \in A \mid a > \lambda\} \in D$ and then $\mathrm{cf}(\Pi A/D) > \lambda$. Thus, if $\langle f_\xi \mid \xi \in \lambda\rangle$ is a universal sequence for λ, we may assume that $f_\xi(a) = \xi$ for all $a \in A \setminus \lambda$.

4.3 Exercise. If $\lambda = \max\ \mathrm{pcf}(A)$, then any universal sequence for λ is cofinal in $\Pi A/J_{<\lambda}$

Universal sequences can be used to prove the following

4.4 Theorem. *For every progressive set A of regular cardinals,*

$$\mathrm{cf}(\Pi A, <) = \max\ \mathrm{pcf}(A).$$

Hence $\mathrm{cf}(\Pi A, <)$ *is a regular cardinal.*

Proof. The partial ordering $<$ in this theorem refers to the everywhere dominance relation on ΠA. The required equality is obtained by first proving $\geq$ and then $\leq$.

Suppose that $\lambda = \max\ \mathrm{pcf}(A)$, and D is an ultrafilter over A such that $\lambda = \mathrm{cf}(\Pi A/D)$. Then $<_D$ extends $<$ on ΠA. That is, for $f, g \in \Pi A$, $f < g$ implies $f <_D g$. This shows that any cofinal set in $(\Pi A, <)$ is also cofinal in $(\Pi A, <_D)$, and hence that $\mathrm{cf}(\Pi A, <) \geq \mathrm{cf}(\Pi A, <_D) = \lambda$.

Now we must exhibit a cofinal subset of $(\Pi A, <)$ of cardinality λ in order to conclude the proof.

Fix for every $\mu \in \mathrm{pcf}(A)$ a universal sequence $f^\mu = \langle f^\mu_i \mid i < \mu\rangle$ for μ. Let F be the set of all functions of the form

$$\sup\{f^{\mu_1}_{i_1}, f^{\mu_2}_{i_2}, \ldots, f^{\mu_n}_{i_n}\}$$

where $\mu_1, \mu_2, \ldots, \mu_n$ is a finite sequence of cardinals in $\mathrm{pcf}(A)$ (with possible repetitions) and $i_k < \mu_k$ are arbitrary indices. (Recall the definition of $\sup\{g_1, \ldots, g_n\}$: at every $a \in A$ it returns $\max\{g_1(a), \ldots, g_n(a)\}$). Clearly $|F| = \lambda$.

4.5 Claim. *F is cofinal in* $(\Pi A, <)$.

Proof of claim. Let $g \in \Pi A$ be any function there. Consider the following collection of subsets of A:

$$I = \{>(f, g) \mid f \in F\}.$$

(Recall that $>(f, g) = \{a \in A \mid f(a) > g(a)\}$.) This collection is closed under unions, that is

$$>(f_1, g)\ \cup\ >(f_2, g)\ =\ >(\sup\{f_1, f_2\}, g).$$

If $A \in I$, namely if $>(f,g) = A$ for some $f \in F$, then evidently $g < f$ as required. But otherwise we obtain a contradiction by extending I to a proper maximal ideal J, and considering $\mu = \mathrm{cf}(\Pi A/J)$. Then f^μ, the universal sequence for μ, is cofinal in $\Pi A/J$, and at the same time it is $\leq_J$ bounded by g since $f \leq_I g$ for all $f \in F$. Yet this is obviously impossible, and thus the theorem is proved. $\dashv$

If $f' = \langle f'_\xi \mid \xi < \lambda \rangle$ is universal sequence for λ, and if $f = \langle f_\xi \mid \xi < \lambda \rangle$ is another sequence in ΠA, $<_{J_{<\lambda}}$-increasing and dominating f' (for all $\xi' < \lambda$ there is $\xi < \lambda$ such that $f'_{\xi'} \leq_{J_{<\lambda}} f_\xi$) then clearly f is also universal for λ. Hence we can use Theorem 2.21 and deduce the following

4.6 Lemma. *Suppose that A is a progressive set of regular cardinals, and $\lambda \in \mathrm{pcf}(A)$. Let μ be the least ordinal such that $A \cap \mu \notin J_{<\lambda}[A]$. Then there is a universal sequence for λ that satisfies $(*)_\kappa$ with respect to $J_{<\lambda}[A]$ for every regular cardinal κ such that $\kappa < \mu$, and in particular for $\kappa = |A|^+$.*

Proof. Observe first that $\mu \leq \lambda + 1$. (Let D be an ultrafilter over A such that $\lambda = \mathrm{cf}(\Pi A/D)$. Then $A \cap (\lambda+1) \in D$, or else $\{a \in A \mid a > \lambda\} \in D$ and then $\mathrm{cf}(\Pi A/D) > \lambda$. Thus $\lambda \in \mathrm{pcf}(A \cap (\lambda+1))$). Observe also that $\mu = \lambda$ is impossible, since λ is regular and $A \cap \lambda$ is necessarily bounded in λ as $|A| < \min A \leq \lambda$. The case $\mu = \lambda + 1$ is rather trivial: $\lambda \in A$ and $J_{<\lambda}[A] = \mathcal{P}(A \cap \lambda)$. In this case the functions defined by $f_\xi(a) = \xi$ for all $a \in A \setminus \lambda$ are as required (and $(*)_\lambda$ holds). So we assume that $\mu < \lambda$ and $A \cap \mu$ is unbounded in μ.

Let $\langle f'_\xi \mid \xi < \lambda \rangle$ be any universal sequence for λ. Theorem 2.21 can be applied to this sequence and to $I = J_{<\lambda}$. This gives a sequence $f_\xi \in \Pi A$ that dominates f'_ξ and that satisfies $(*)_\kappa$ for every regular cardinal κ such that $\kappa^{++} < \lambda$ and $\{a \in A \mid a \leq \kappa^{++}\} \in I$. Thus $(*)_\kappa$ holds for every regular $\kappa < \mu$. $\dashv$

We intend to prove next the existence of a generating set for $J_{<\lambda^+}$. For this we need first the following characterization of generators for $J_{<\lambda^+}$.

4.7 Lemma. *If A is a progressive set of regular cardinals and $B \subseteq A$ is any subset, then*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B \tag{I.18}$$

if and only if

$$B \in J_{<\lambda^+}[A] \text{ and} \tag{I.19}$$

$$\text{If } D \text{ is any ultrafilter over } A \text{ with } \mathrm{cf}(\Pi A/D) = \lambda \text{ then } B \in D. \tag{I.20}$$

Proof. Assume first that (I.18) holds. Then (I.19) is obvious. We prove (I.20). If D is any ultrafilter over A with $\mathrm{cf}(\Pi A/D) = \lambda$, then $D \cap J_{<\lambda^+}[A] \neq \emptyset$, and if $X \in D \cap J_{<\lambda^+}[A]$ is any set in the intersection then (I.18) implies that $X \setminus B \in J_{<\lambda}[A]$. Since $D \cap J_{<\lambda} = \emptyset$, $B \in D$ follows.

Now assume that (I.19) and (I.20) hold, and we prove that $J_{<\lambda^+}[A] = J_{<\lambda}[A] + B$.

Since $B \in J_{<\lambda^+}[A]$, $J_{<\lambda^+}[A] \supseteq J_{<\lambda}[A] + B$.

To prove $J_{<\lambda^+}[A] \subseteq J_{<\lambda}[A] + B$ assume $X \in J_{<\lambda^+}[A]$ and prove that $X \setminus B \in J_{<\lambda}$ as follows. Let D be any ultrafilter over A such that $X \setminus B \in D$. Since $X \in J_{<\lambda^+}$, $\mathrm{cf}(\Pi A/D) < \lambda^+$. But $\mathrm{cf}(\Pi A/D) = \lambda$ is impossible as $B \notin D$ and we assume (I.20). Hence $\mathrm{cf}(\Pi A/D) < \lambda$. ⊣

4.8 Theorem (Normality). *If A is a progressive set of regular cardinals, then every cardinal $\lambda \in \mathrm{pcf}(A)$ is normal: there exists a set $B_\lambda \subseteq A$ such that*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda.$$

Proof. By Lemma 4.6, there exists a universal sequence $f = \langle f_\xi \mid \xi < \lambda \rangle$ for λ that satisfies $(*)_{|A|^+}$. Hence f has an exact upper bound h in $\mathrm{On}^A/J_{<\lambda}$. Since the identity function is an upper bound of f, we can assume that $h(a) \leq a$ for every $a \in A$. Now define

$$B = \{a \in A \mid h(a) = a\}.$$

We are going to prove that B satisfies the two properties (I.19) and (I.20) which concludes the theorem and shows that B is a generator for $J_{<\lambda^+}$. We first prove that $B \in J_{<\lambda^+}[A]$. If D is any ultrafilter over A containing B then

$$\mathrm{cf}(\Pi A/D) \leq \lambda \tag{I.21}$$

is deduced in two steps. If $D \cap J_{<\lambda} \neq \emptyset$, then the strict inequality of (I.21) holds by definition of $J_{<\lambda}$. But if $D \cap J_{<\lambda} = \emptyset$, then h remains the exact upper bound of the $<_D$ increasing sequence f in $<_D$ (just because D extends the dual filter of $J_{<\lambda}$). So $\mathrm{cf}(\Pi h/D) = \lambda$. As h is $=_D$ equivalent to the identity function over A, $\Pi A/D$ has cofinality λ.

To prove (I.20), suppose that D is an ultrafilter over A and $\mathrm{cf}(\Pi A/D) = \lambda$. If $B \notin D$ then $\{a \in A \mid h(a) < a\} \in D$, and thus $[h]_D$ (the $=_D$-equivalence class of h) is in $\Pi A/D$. Yet $D \cap J_{<\lambda}[A] = \emptyset$ (or else $\mathrm{cf}(\Pi A/D) < \lambda$), and this implies that $f_\xi <_D h$ for every $\xi < \lambda$ (because $f_\xi <_{J_{<\lambda}} h$). So f is *not* cofinal in $\Pi A/D$, in contradiction to f being a universal sequence for λ. ⊣

The generator set B_λ is not uniquely determined, but if B_1 and B_2 are two generators (they both satisfy I.18), then the symmetric difference $B_1 \triangle B_2$ is in $J_{<\lambda}[A]$. So generators *are* uniquely determined modulo $J_{<\lambda}$, and we can use a "generic" notation.

4.9 Notation. For a progressive set of regular cardinals A and for any cardinal $\lambda \in \mathrm{pcf}(A)$, $B_\lambda[A]$ denotes a subset $B \subseteq A$ such that (I.19) and (I.20) hold, or equivalently

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B. \tag{I.22}$$

We also use the expression "B is a $B_\lambda[A]$ set" if (I.22) holds for B. We often write B_λ instead of $B_\lambda[A]$, when the identity of A is obvious.

The sequence $\langle B_\lambda[A] \mid \lambda \in \mathrm{pcf}(A)\rangle$ is called a "generating sequence" for A, because the ideal $J_{<\lambda}$ is generated by the collection $\{B_{\lambda_0} \mid \lambda_0 < \lambda\}$ (see Corollary 4.12). It is convenient to write $B_\lambda = \emptyset$ when $\lambda \notin \mathrm{pcf}(A)$.

The following conclusion will be needed later on.

4.10 Lemma. *Suppose that A is a progressive set of regular cardinals. If $A_0 \subseteq A$ and $\lambda \in \mathrm{pcf}(A_0)$, then $B_\lambda[A_0] =_{J_{<\lambda}[A_0]} A_0 \cap B_\lambda[A]$. (This justifies our inclination to write B_λ instead of $B_\lambda[A_0]$.)*

Proof. We prove (I.19) and (I.20) for $A_0 \cap B_\lambda[A]$. Clearly $A_0 \cap B_\lambda[A] \in J_{\leq\lambda}[A_0]$. If D_0 is any ultrafilter over A_0 such that $\mathrm{cf}(\Pi A_0/D_0) = \lambda$, then $A_0 \cap B_\lambda[A] \in D_0$ can be argued as followed. Assume $A_0 \setminus B_\lambda[A] \in D_0$, and extend D_0 to an ultrafilter over A, still denoted D_0. Then $\mathrm{cf}(\Pi A/D_0) = \lambda$ and $B_\lambda[A] \notin D_0$ is in contradiction to (I.20). $\dashv$

For a progressive set A with $\lambda = \max\ \mathrm{pcf}(A)$ and B a $B_\lambda[A]$ set, we have by (I.20) that

$$A \setminus B \in J_{<\lambda} \tag{I.23}$$

since $A \in J_{<\lambda^+}[A]$. Hence we can take $B_{\max(\mathrm{pcf}(A))} = A$.

We will conclude that the ideal $J_{<\lambda}[A]$ is (finitely) generated by the sets $\{B_\mu[A] \mid \mu < \lambda\}$ using the following "compactness" theorem, which says that any set $X \in J_{<\lambda}$ is covered by a finite collection of B_μ's for $\mu < \lambda$.

4.11 Theorem. *(Compactness) Suppose that A is a progressive set of regular cardinals and $\langle B_\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ is a generating sequence for A, then for any $X \subseteq A$*

$$X \subseteq B_{\lambda_1} \cup B_{\lambda_2} \ldots \cup B_{\lambda_n}$$

for some finite set $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathrm{pcf}(X)$.

Proof. This can be proved by induction on $\lambda = \max\ \mathrm{pcf}(X)$, since $X \setminus B_\lambda \in J_{<\lambda}$ and so $\max\ \mathrm{pcf}(X \setminus B_\lambda) < \lambda$. $\dashv$

4.12 Corollary. *If A is a progressive set of regular cardinals then for every cardinal λ, for every set $X \subseteq A$, $X \in J_{<\lambda}[A]$ iff X is included in a finite union of sets from $\{B_{\lambda'} \mid \lambda' < \lambda\}$.*

Observe that $\lambda \notin \mathrm{pcf}(A \setminus B_\lambda[A])$. For let D_0 be any ultrafilter over $A_0 = A \setminus B_\lambda[A]$. Extend D_0 to an ultrafilter D over A. Since $\Pi A_0/D_0$ is isomorphic to $\Pi A/D$, it suffices to prove that $\mathrm{cf}(\Pi A/D) \neq \lambda$. But since A_0 is disjoint to $B_\lambda[A]$, $B_\lambda[A] \notin D$. So (I.20) implies this, and we have obtained the following result. A set $B \in J_{<\lambda^+}[A]$ is a B_λ set if and only if $\lambda \notin \mathrm{pcf}(A \setminus B)$.

If $\lambda \in \mathrm{pcf}(A)$ and $f = \langle f_\xi \mid \xi < \lambda \rangle$ is a universal sequence for λ, then the definition of $B_\lambda[A] = \{a \in A \mid h(a) = a\}$, where h is an exact upper bound of f, shows that $\langle f_\xi \restriction B_\lambda \mid \xi < \lambda \rangle$ is cofinal in $\Pi B_\lambda / J_{<\lambda}$. This result is sufficiently interesting to be isolated as a theorem (and we give a somewhat different proof).

4.13 Theorem. *If A is a progressive set of regular cardinals and $\lambda \in \mathrm{pcf}(A)$, then for some set $B \subseteq A$ we have $\mathrm{tcf}(\Pi B/J_{<\lambda}[B]) = \lambda$. In fact, any universal sequence for λ is cofinal in $\Pi B_\lambda / J_{<\lambda}$ and thus shows that*

$$\mathrm{tcf}(\Pi B_\lambda / J_{<\lambda}) = \lambda. \tag{I.24}$$

Proof. We know that there exists a universal sequence for λ and that there exists a generating set B_λ. We will prove that any universal sequence $f = \langle f_\xi \mid \xi < \lambda \rangle$ for λ is cofinal in $\Pi B_\lambda / J_{<\lambda}$. That is, if $h \in \Pi B_\lambda$ is any function then

$$\leq(f_\xi \restriction B_\lambda, h) \in J_{<\lambda} \text{ for some } \xi < \lambda.$$

Otherwise the sets $\leq(f_\xi \restriction B_\lambda, h)$ are positive and decreasing with $\xi < \lambda$ (mod $J_{<\lambda}$). Hence there is a filter over B_λ containing them all and extending the dual filter of $J_{<\lambda}[B_\lambda]$. Extending this filter to an ultrafilter D over A, $D \cap J_{<\lambda}[A] = \emptyset$ and the ultraproduct $\Pi A/D$ has cofinality λ (as $B_\lambda \in D$ and $D \cap J_{<\lambda} = \emptyset$). In this ultrapower h bounds all functions in f, in contradiction to the assumption that f is universally cofinal. Thus the restriction to $B_\lambda[A]$ of any universal sequence for λ is cofinal in $\Pi B_\lambda / J_{<\lambda}$. ⊣

In particular, (I.24) shows (again) that $\lambda = \max\ \mathrm{pcf}(B_\lambda)$ whenever $\lambda \in \mathrm{pcf}(A)$. We have, more generally, the following characterization.

4.14 Lemma. *The following are equivalent for every filter F over a progressive set of regular cardinals A and for every cardinal λ.*

1. $\mathrm{tcf}(\Pi A/F) = \lambda$.
2. *$B_\lambda \in F$, and F contains the dual filter of $J_{<\lambda}[A]$.*
3. *$\mathrm{cf}(\Pi A/D) = \lambda$ for every ultrafilter D that extends F.*

In particular we get for every ultrafilter D that

$$\mathrm{cf}(\Pi A/D) = \lambda \textit{ iff } B_\lambda \in D \textit{ and } D \cap J_{<\lambda} = \emptyset. \tag{I.25}$$

Equivalently,

$$\mathrm{cf}(\Pi A/D) = \lambda \textit{ iff } \lambda \textit{ is the least cardinal such that } B_\lambda \in D. \tag{I.26}$$

Proof. Fix a filter F and a cardinal λ. $1 \Longrightarrow 3$ is obvious. Assume 3 and we prove 2. Since $\mathrm{cf}(\Pi A/D) = \lambda$ for every ultrafilter D that extends F, $B_\lambda \in D$ for every such ultrafilter (by (I.20)). Hence $B_\lambda \in F$. It is clear that F contains the dual filter of $J_{<\lambda}$, or else an extension of F can be found that intersects $J_{<\lambda}$ and thus has an ultraproduct with cofinality below λ.

Assume now 2, and then the fact already proved that $\mathrm{tcf}(\Pi B_\lambda/J_{<\lambda}) = \lambda$ shows that $\mathrm{tcf}(\Pi A/F) = \lambda$ (as $\Pi A/F$ and $\Pi B_\lambda/F$ are isomorphic, since $B_\lambda \in F$).

In particular, if D is an ultrafilter over A, then $D \cap J_{<\lambda} = \emptyset$ iff the dual filter of $J_{<\lambda}$ is contained in D. So the equivalence of 1 and 2 of the lemma estabilshes (I.25). ⊣

4.15 Exercise. 1. If D is an ultrafilter over a progressive set A, and λ is the least cardinal such that $B_\lambda \in D$, then $\lambda = \mathrm{cf}(\Pi A/D)$.

2. If A is a progressive set of regular cardinals and $E = \mathrm{pcf}(A)$ is also progressive, then

$$\mathrm{pcf}(B_\lambda[A]) =_{J_{<\lambda}[E]} B_\lambda[\mathrm{pcf}(A)].$$

(Use Theorem 3.12.)

4.16 Exercise. If A is a progressive set of regular cardinals then for every cardinal λ, $\lambda = \max\ \mathrm{pcf}(A)$ iff $\lambda = \mathrm{tcf}(\Pi A/J_{<\lambda})$ iff $\lambda = \mathrm{cf}(\Pi A/J_{<\lambda})$.

In Theorem 2.23 we have proved for μ, a singular cardinal with uncountable cofinality, that $\mu^+ = \mathrm{tcf}(\Pi C^{(+)}/J^{bd})$ for some closed unbounded set of cardinals $C \subset \mu$. Since $J_{<\mu} = J_{<\mu^+} \subseteq J^{bd}$, an apparently stronger claim is obtained by asserting $\mathrm{tcf}(\Pi C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$.

4.17 Exercise (The Representation Theorem). If μ is a singular cardinal with uncountable cofinality, then for some closed unbounded set of cardinals $C \subseteq \mu$, $\mathrm{tcf}(\Pi C^{(+)}/J_{<\mu}[C^{(+)}]) = \mu^+$. Thus $\mu^+ = \max\ \mathrm{pcf}\ C^{(+)}$.

Hint. Let $C_0 \subseteq \mu$ be a closed unbounded set of limit cardinals such that $\mu^+ = \mathrm{tcf}(\Pi C_0^{(+)}/J^{bd})$. Then define $C \subseteq C_0$ so that $C^{(+)} = B_{\mu^+}[C_0^{(+)}]$. Prove that $C_0 \setminus C$ is bounded in μ. Then use Theorem 4.13.

4.18 Exercise. For any filter F over a progressive set A of regular cardinals, define

$$\mathrm{pcf}_F(A) = \{\mathrm{cf}(\Pi A/D) \mid D \text{ an ultrafilter over } A \text{ that extends } F\}.$$

1. Prove that max $\mathrm{pcf}_F(A)$ exists.

   Hint. Look at the minimal λ such that $F \cap J_{\leq\lambda} \neq \emptyset$.

2. Deduce that $\mathrm{cf}(\Pi A/F) = \max \mathrm{pcf}_F(A)$, so that the cofinality of this partial ordering is a regular cardinal.

3. If $B \subseteq \mathrm{pcf}_F(A)$ is progressive, then $\mathrm{pcf}(B) \subseteq \mathrm{pcf}_F(A)$.

4. Suppose that A is a progressive *interval* of regular cardinals, and let F be the filter of co-bounded subsets of A ($X \in F$ iff $A \setminus X$ is bounded in A). Then $\mathrm{pcf}_F(A)$ is an interval of regular cardinals.

5. The cofinality of $[\mu]^\kappa$

Some of the most important applications of the pcf theory will be described in this section. For example, we will prove that $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$. For this result we investigate obedient universal sequences and their relationship with characteristic functions of elementary substructures. Some of the theorems about obedient sequences proved and used in this section will be applied in the following section to "elevated" sequences. These sequences are not obedient, but they share enough properties with the obedient sequences to enable uniform proofs. This explains our desire to deal here with the shared properties (I.32 and I.33) rather than with obedient sequences.

As usual, A is a progressive set of regular cardinals. Recall how $B_\lambda[A]$ was obtained. First a universal sequence $\langle f_\xi \mid \xi < \lambda \rangle$ for λ was defined which satisfied $(*)_{|A|^+}$, then an exact upper bound h was constructed, and finally the set $B_\lambda = \{a \in A \mid h(a) = a\}$ was shown to generate $J_{<\lambda^+}[A]$ over $J_{<\lambda}[A]$. Once this is done, we have greater flexibility in tuning-up B_λ by using elementary substructures, and we therefore say first a few words about these structures.

5.1. Elementary substructures

Elementary substructures are extensively used in the pcf theory and its applications, and in this section we study some basic properties of their characteristic functions.

Let Ψ be a "sufficiently large" cardinal, and H_Ψ be the $\in$-structure whose universe is the collection H_Ψ of all sets hereditarily of cardinality less than Ψ (which means having transitive closure of size $< \Psi$). The expression "sufficiently large" depends on the context and means that Ψ is regular and is sufficiently large to include in H_Ψ all sets that were discussed so far. We also add to the structure H_Ψ a well-ordering $<^*$ of its universe. We shall seldom mention $<^*$ explicitly, but it allows us to assume that the objects we talk about are uniquely determined.

For the rest of this section κ denotes a regular cardinal such that $|A| < \kappa < \min(A)$.

5.1 Definition. An increasing and continuous chain of length λ of elementary substructures of H_Ψ is a sequence $\langle M_i \mid i < \lambda \rangle$ such that

1. Each M_i is an elementary substructure of H_Ψ,
2. $i_1 < i_2 < \lambda$ implies that $M_{i_1} \subset M_{i_2}$, and
3. for every limit ordinal $\delta < \lambda$, $M_\delta = \bigcup_{i<\delta} M_i$ (this is continuity).

We say in this paper that an elementary substructure $M \prec H_\Psi$ is "κ-presentable" if and only if $M = \bigcup_{i<\kappa} M_i$ where $\langle M_i \mid i < \kappa \rangle$ is an increasing and continuous chain of length κ such that

1. M has cardinality κ and $\kappa + 1 \subset M$.
2. For every $i < \kappa$, $M_i \in M_{i+1}$. (Thus $M_i \in M_j$ for $i < j$.)

We do not make any assumption on the cardinality of M_i for $i < \kappa$, which may be κ or smaller than κ.

In order to define a κ-presentable elementary substructure define, recursively, the approaching structures M_α, and observe that each M_α (and even the sequence $\langle M_\alpha \mid \alpha \leq \beta \rangle$) is an element of H_Ψ and thus can be incorporated in $M_{\beta+1} \prec H_\Psi$.

We shall use the following observation. Let $\bar{M}_\alpha$ denote the ordinal closure of $M_\alpha \cap \mathrm{On}$. That is $\gamma \in \bar{M}_\alpha$ iff $\gamma \in M_\alpha \cap \mathrm{On}$ or γ is a limit of ordinals in M_α. Since $M_\alpha \in M_{\alpha+1}$ and $M_\alpha \subset M_{\alpha+1}$, $\bar{M}_\alpha \in M_{\alpha+1}$, and $\bar{M}_\alpha \subseteq M_{\alpha+1}$

For any structure N, we let Ch_N be the "characteristic function" of N. That is, the function defined on any regular cardinal μ such that $\|N\| < \mu$ by

$$Ch_N(\mu) = \sup\ N \cap \mu.$$

Then $Ch_N(\mu) \in \mu$ since μ is regular and N is of smaller cardinality.

A very useful fact that we are going to prove is that if M is κ-presentable, then for cardinals $\lambda < \mu$, $M \cap \mu$ can be reconstructed from $M \cap \lambda$ and the characteristic function of M restricted to the successor cardinals in the interval $(\lambda, \mu]$. We shall use the following form of this fact.

5.2 Theorem. *Suppose that M and N are elementary substructures of H_Ψ. Let $\kappa < \mu$ be any cardinals (κ is always regular uncountable).*

1. *If $M \cap \kappa \subseteq N \cap \kappa$, and, for every successor cardinal $\alpha^+ \in M \cap \mu + 1$,*

$$\sup\ M \cap \alpha^+ = \sup\ M \cap N \cap \alpha^+, \tag{I.27}$$

then $M \cap \mu \subseteq N \cap \mu$.

2. *Therefore, if* M *and* N *are both* κ*-presentable and for every successor cardinal* $\alpha^+ \in \mu + 1$

$$\sup\ M \cap \alpha^+ = \sup\ N \cap \alpha^+, \tag{I.28}$$

then $M \cap \mu = N \cap \mu$.

Proof. This is a bootstrapping argument. We prove by induction on δ, a cardinal in the interval $[\kappa, \mu]$, that $M \cap \delta \subseteq N \cap \delta$. For $\delta = \kappa$ this is an assumption. If δ is a limit cardinal, then $M \cap \delta \subseteq N \cap \delta$ is an immediate application of the inductive assumption that $M \cap \delta' \subseteq N \cap \delta'$ for every cardinal δ' in the interval $[\kappa, \delta)$. Assume now that $M \cap \delta \subseteq N \cap \delta$, and we shall argue for $M \cap \delta^+ \subseteq N \cap \delta^+$. If $\delta^+ \notin M$, then M contains no ordinals from the interval $[\delta, \delta^+]$ and the claim is obvious. So assume that $\delta^+ \in M$. (And hence $\delta^+ \in N$ since $[\delta, \delta^+] \cap N \neq \emptyset$.) Let $\gamma = \sup(M \cap \delta^+) = \sup(M \cap N \cap \delta^+)$. Now if $\alpha \in M \cap \gamma$, then there exists some $\beta \in M \cap N \cap \gamma$ such that $\alpha < \beta$. Consider the structure $(H_\Psi, \in, <^*)$ of which M and N are elementary substructures, and pick an injection $f : \beta \to \delta$ that is minimal with respect to the well-ordering $<^*$ of H_Ψ. Then $f \in M \cap N$ because f is definable from the parameter β. Since $\alpha \in M$, $f(\alpha) \in M$, and hence $f(\alpha) \in N$. But then, applying f^{-1} in N, we get $\alpha \in N$. Thus $M \cap \beta \subseteq N \cap \beta$.

For the second part of the theorem, let $M = \bigcup_{\xi<\kappa} M_\xi$ and $N = \bigcup_{\xi<\kappa} N_\xi$ be presentations for M and N. Observe that $M \cap \kappa = N \cap \kappa = \kappa$. Let α^+ be any successor cardinal in the interval $(\kappa, \mu]$. We assume that $\gamma = Ch_M(\alpha^+) = Ch_N(\alpha^+)$. We claim that there is a subset of $M \cap N$ that is closed and unbounded in γ. Indeed, the approaching substructures M_ξ provide a closed unbounded sequence $\langle \sup(M_\xi \cap \alpha^+) \mid \xi \in \kappa \rangle$ which is cofinal in γ. Likewise, N contains a closed unbounded sequence of order-type κ cofinal in γ. The intersection of these closed unbounded sets is as required. Hence $\sup\ M \cap \alpha^+ = \sup\ N \cap \alpha^+ = \sup\ M \cap N \cap \alpha^+$ holds and $M \cap \mu = N \cap \mu$ is obtained by the first part of the theorem. ⊣

Recall that a sequence of functions in ΠA is universal for λ if it is $J_{<\lambda}$ increasing and cofinal in $\Pi A/D$ whenever $\mathrm{cf}(\Pi A/D) = \lambda$. Recall also (Equation I.3) that a sequence $\langle p_\xi \mid \xi < \lambda \rangle$ of members of a partial ordering $(P, <_P)$ is persistently cofinal iff every member of P is dominated by all members of the sequence with a sufficiently large index.

5.3 Definition. We say that a sequence $\langle f_\xi \mid \xi < \lambda \rangle$ of functions in ΠA is *persistently cofinal for* λ if their restrictions to B_λ form a persistently cofinal sequence in $\Pi B_\lambda / J_{<\lambda}$. Namely if for every $h \in \Pi A$ there exists $\xi_0 < \lambda$ such that

$$h \restriction B_\lambda <_{J_{<\lambda}} f_\xi \restriction B_\lambda$$

for all $\xi_0 \leq \xi < \lambda$ (where $B_\lambda = B_\lambda[A]$).

For example, if $\langle f_\xi \mid \xi < \lambda\rangle$ is universal for λ then it is persistently cofinal (see Theorem 4.13), and if the functions F_ξ are such that $f_\xi \leq_{J_{<\lambda}} F_\xi$ for all $\xi < \lambda$, then $\langle F_\xi \mid \xi < \lambda\rangle$ is also persistently cofinal, although it is not necessarily $J_{<\lambda}$ increasing. Clearly, an arbitrary λ sequence in ΠA is universal for λ iff it is $J_{<\lambda}$ increasing and persistently cofinal.

A basic observation which is used later to define the transitive generators is the following.

5.4 Lemma. *Suppose that the progressive set A and the cardinal $\lambda \in \mathrm{pcf}(A)$ belong to an elementary substructure $N \prec H_\Psi$ so that $N = \bigcup_{\alpha<\kappa} N_\alpha$ where $|A| < \kappa < \min(A)$ is a regular cardinal, $|N| = \kappa$, $\kappa+1 \subset N$, and $\langle N_\alpha \mid \alpha < \kappa\rangle$ is an increasing chain of elementary substructures of H_Ψ. If a sequence of functions $f = \langle f_\xi \mid \xi < \lambda\rangle \in N$, with $f_\xi \in \Pi A$, is persistently cofinal for λ, then for every $\xi \geq \sup(N \cap \lambda)$*

$$\leq(Ch_N, f_\xi) = \{a \in A \mid Ch_N(a) \leq f_\xi(a)\} \text{ is a } B_\lambda[A] \text{ set.} \tag{I.29}$$

Proof. We first make some preliminary observations. Since $\kappa < \min A$, we have that $Ch_N \restriction A \in \Pi A$. Since $A, \lambda \in N = \bigcup_{\alpha<\kappa} N_\alpha$, we may as well assume that $A, \lambda, f \in N_0$ (or else re-enumerate the structures). Since $|A| < \kappa$ and $\kappa \subset N$, $A \subset N$ and we can assume that $A \subset N_0$. Since Ψ is sufficiently large, all the pcf theory involved in defining $B_\lambda[A]$ etc. can be done in H_Ψ and hence in N_0. We may assume again that a generating set $B = B_\lambda[A]$ is in N_0. Suppose that $\xi \geq \sup(N \cap \lambda)$. To prove (I.29) we need two inclusions:

1. $Ch_N \restriction B \ \leq_{J_{<\lambda}} f_\xi \restriction B$, which shows that $B \ \subseteq_{J_{<\lambda}} \leq(Ch_N, f_\xi)$.
2. $\leq(Ch_N, f_\xi) \cap (A \setminus B) \in J_{<\lambda}$, which shows that $\leq(Ch_N, f_\xi) \ \subseteq_{J_{<\lambda}} \ B$.

We prove 1. For every $a \in A$, if $f_\xi(a) < Ch_N(a)$ then there exists an index $\alpha = \alpha(a) < \kappa$ such that $f_\xi(a) < Ch_{N_\alpha}(a)$. Since $|A| < \kappa$ there exists a single index $\alpha < \kappa$ such that, for every $a \in A$, $f_\xi(a) < Ch_N(a)$ implies that $f_\xi(a) < Ch_{N_\alpha}(a)$. Hence for every $a \in A$

$$f_\xi(a) < Ch_N(a) \text{ iff } f_\xi(a) < Ch_{N_\alpha}(a). \tag{I.30}$$

But the sequence f is persistently cofinal in $\Pi B/J_{<\lambda}$, and hence $h \restriction B <_{J_{<\lambda}} f_\xi \restriction B$ for every $h \in N \cap \Pi A$, because $\xi \geq \sup(N \cap \lambda)$. In particular, for $h = Ch_{N_\alpha} \in N$, we get

$$Ch_{N_\alpha} \restriction B \leq_{J_{<\lambda}} f_\xi \restriction B \text{ (in fact } <_{J_{<\lambda}}).$$

That is, $\{b \in B \mid f_\xi(b) < Ch_{N_\alpha}(b)\} \in J_{<\lambda}$. Hence, by (I.30), $\{b \in B \mid f_\xi(b) < Ch_N(b)\} \in J_{<\lambda}$. Thus

$$Ch_N \restriction B \leq_{J_{<\lambda}} f_\xi \restriction B.$$

This proves 1.

Now we prove 2. That is

$$\{a \in A \setminus B \mid Ch_N(a) \le f_\xi(a)\} \in J_{<\lambda}. \tag{I.31}$$

As $\lambda \notin \mathrm{pcf}(A \setminus B)$, $J_{<\lambda}[A \setminus B] = J_{<\lambda^+}[A \setminus B]$ and hence $\Pi(A \setminus B)/J_{<\lambda}$ is λ^+-directed, and f (with functions restricted to $A \setminus B$) has an upper bound. Since $f \in N$, we have this upper bound in N. Let $h \in N \cap \Pi(A \setminus B)$ be an upper bound in $<_{J_{<\lambda}}$ of the sequence f restricted to $A \setminus B$. Then

$$f_\xi \restriction (A \setminus B) \ <_{J_{<\lambda}} \ h \ < \ Ch_N \restriction (A \setminus B).$$

But this is exactly (I.31) $\dashv$

5.2. Minimally obedient sequences

Suppose that δ is a limit ordinal and $f = \langle f_\xi \mid \xi < \delta \rangle$ is a sequence of functions $f_\xi \in \Pi A$, where A is a set of regular cardinals and $|A|^+ \le \mathrm{cf}(\delta) < \min(A)$ holds. For every closed unbounded set $E \subseteq \delta$ of order type $\mathrm{cf}(\delta)$ let

$$h_E = \sup\{f_\xi \mid \xi \in E\}.$$

That is, $h_E(a) = \sup\{f_\xi(a) \mid \xi \in E\}$. Since $\mathrm{cf}(\delta) < \min(A)$, $h_E \in \Pi A$. We say that h_E is the "supremum along E of the sequence f". Observe that if $E_1 \subseteq E_2$ then $h_{E_1} \le h_{E_2}$. The following lemma says that among all functions obtained as suprema along closed unbounded subsets of δ there is a minimal one in the $\le$ ordering.

5.5 Lemma. *Let δ and f be as above (so $|A| < \mathrm{cf}(\delta) < \min(A)$ and f is a sequence of length δ of functions in ΠA). There is a closed unbounded set $C \subseteq \delta$ of order type $\mathrm{cf}(\delta)$ such that*

$$h_C(a) \le h_E(a)$$

for every $a \in A$ and $E \subseteq \delta$ closed and unbounded (of order type $\mathrm{cf}(\delta)$).

Proof. Assume that there is no such closed unbounded set $C \subseteq \delta$ as required. We construct a decreasing sequence $\langle E_\alpha \mid \alpha < |A|^+ \rangle$ of closed unbounded subsets of δ of order type $\mathrm{cf}(\delta)$ each, such that for every $\alpha < |A|^+$, $h_{E_\alpha} \not\le h_{E_{\alpha+1}}$. (Since $|A| < \mathrm{cf}(\delta)$, at limit stages of the construction we may take the intersection of the clubs so far constructed.) Then find a single $a \in A$ such that $h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)$ for an unbounded set of indices α. Yet this is obviously impossible. $\dashv$

In applications of this lemma, an ideal J over A is assumed and the sequence $\langle f_\xi \mid \xi < \delta \rangle$ is $<_J$-increasing. In that case, the minimal function $f_C = \sup\{f_\xi \mid \xi \in C\}$ $\le$-bounds each f_ξ, for $\xi \in C$, and hence $\le_J$-bounds all f_ξ's for $\xi < \delta$. This function f_C is called "minimal club-obedient bound of $\langle f_\xi \mid \xi < \delta \rangle$".

5.6 Definition (Minimally Obedient Universal Sequence). Suppose that λ is in $\mathrm{pcf}(A)$ and $f = \langle f_\xi \mid \xi < \lambda \rangle$ is a universal sequence for λ. Let κ be a fixed regular cardinal such that $|A| < \kappa < \min(A)$. We say that f is a "minimally obedient (at cofinality κ)" if for every $\delta < \lambda$ such that $\mathrm{cf}(\delta) = \kappa$, f_δ is the minimal club-obedient bound of $\langle f_\xi \mid \xi < \delta \rangle$.

The universal sequence f is said to be "minimally obedient" if $|A|^+ < \min(A)$ and it is minimally obedient for every regular κ such that $|A| < \kappa < \min(A)$.

Suppose that $|A|^+ < \min(A)$ and $\lambda \in \mathrm{pcf}(A)$. In order to arrange a minimally obedient universal sequence for λ start with an arbitrary universal sequence $\langle f^0_\xi \mid \xi < \lambda \rangle$ and define the functions f_ξ by induction on $\xi < \lambda$ such that:

1. $f_0 = f^0_0$, and $f_{\xi+1}$ is such that
$$\max\{f_\xi, f^0_\xi\} < f_{\xi+1}.$$
2. At limit stages $\delta < \lambda$ with $\mathrm{cf}(\delta) = \kappa$ and such that $|A| < \kappa < \min(A)$ let f_δ be the minimal club-obedient bound of $\langle f_\xi \mid \xi < \delta \rangle$.
3. At limit stages $\delta < \lambda$ with $\mathrm{cf}(\delta)$ not of that form use the fact that $\Pi A/J_{<\lambda}$ is λ-directed to define f_δ as a $<_{J_{<\lambda}}$ bound of $\langle f_\xi \mid \xi < \delta \rangle$.

Minimally obedient sequences will be used in conjunction with κ-presentable elementary substructures.

5.7 Lemma. *Let A be a progressive set of regular cardinals, and κ be a regular cardinal such that $|A| < \kappa < \min(A)$. Suppose that*

1. *$\lambda \in \mathrm{pcf}(A)$,*
2. *$f = \langle f_\xi \mid \xi < \lambda \rangle$ is a minimally obedient at cofinality κ, universal sequence for λ, and*
3. *$N \prec H_\Psi$ (for Ψ sufficiently large) is an elementary, κ-presentable substructure of H_Ψ such that $\lambda, f, A \in N$. (So $A \subset N$.)*

Let $\bar{N}$ denote the ordinal closure of $N \cap \mathrm{On}$, that is the set of ordinals that are in N or that are limit of ordinals in N. Then for every $\gamma \in (\bar{N} \cap \lambda) \setminus N$ there is a closed unbounded set $C \subseteq \gamma \cap N$ (of order-type κ) such that $f_\gamma = \sup\{f_\xi \mid \xi \in C\}$ and thus

$$f_\gamma(a) \in \bar{N} \cap a \text{ for every } a \in A.$$

In particular, for $\gamma = Ch_N(\lambda)$, $\gamma \in \bar{N} \setminus N$, and $f_\gamma = \sup\{f_\xi \mid \xi \in C\}$ for a closed unbounded set $C \subseteq \gamma \cap N$ such that

1. *each* f_ξ *is in* N*, and*

2. *for every* $h \in N \cap \Pi A$

$$h \restriction B_\lambda[A] <_{J_{<\lambda}} f_\xi \restriction B_\lambda[A]$$

for some $\xi \in C$.

Proof. Since N is κ-presentable, $N = \bigcup_{\alpha<\kappa} N_\alpha$ is the union of an increasing and continuous chain such that $N_\alpha \in N_{\alpha+1}$. It follows for every $\gamma \in \bar{N}$, that either $\gamma \in N$ or $\mathrm{cf}(\gamma) = \kappa$. Indeed, if $\gamma \in \bar{N} \setminus N$, then γ is a limit point of ordinals in N and yet γ is not a limit point of ordinals in any N_α (or else $\gamma \in \bar{N}_\alpha \subset N$). Hence $\sup(\gamma \cap N_\alpha) < \gamma$ and

$$E = \{\sup(\gamma \cap N_\alpha) \mid \alpha < \kappa\}$$

is closed unbounded in γ and of order-type κ. Thus $\mathrm{cf}(\gamma) = \kappa$. Observe that $E \subseteq N$, because $N_\alpha \in N$ implies that $\bar{N}_\alpha \subset N$ and in particular $\sup(\gamma \cap N_\alpha) \in N$.

Now take $\gamma \in (\bar{N} \cap \lambda) \setminus N$ and consider $f_\gamma(a)$ for $a \in A$. Since $\mathrm{cf}(\gamma) = \kappa$, f_γ is the minimal club-obedient bound of $\langle f_\xi \mid \xi < \gamma\rangle$, and there is thus a closed unbounded set $C \subseteq \gamma$ such that $f_\gamma = f_C$. It follows from the minimality of f_C that $f_C = f_{C\cap E}$ and we may thus assume at the outset that $C \subseteq N$. So $f_\gamma = f_C = \sup\{f_\xi \mid \xi \in C\}$ is the supremum of a set of functions that are all in N. (As $C \subset N$ implies that $f_\xi \in N$ for $\xi \in C$.) This shows that $f_\gamma(a) \in \bar{N}$ for every $a \in A$.

In particular, if $\gamma = Ch_N(\lambda)$, then $\gamma < \lambda$ because $\kappa < \lambda$ and N has cardinality κ. So $\gamma \in \bar{N} \setminus N$. Item *2* is a consequence of the fact that f is a universal sequence (see Theorem 4.13) and that C is unbounded in $N \cap \lambda$. ⊣

The conclusions of lemmas 5.4 and 5.7 will be given names (I.32 and I.33) so that we can easily refer to these properties in the future. Let A be a progressive set of regular cardinals and suppose that κ is a regular cardinal such that $|A| < \kappa < \min\ A$. Suppose that $\lambda \in \mathrm{pcf}(A)$, and $f = \langle f_\xi \mid \xi \in \lambda\rangle$ is a sequence of functions in ΠA. We shall refer to the following two properties of a κ-presentable $N \prec H_\Psi$ and a sequence $f = \langle f_\xi \mid \xi < \lambda\rangle$ such that $f \in N$.

If $\gamma = Ch_N(\lambda)$, then

$$\{a \in A \mid Ch_N(a) \leq f_\gamma(a)\} \tag{I.32}$$

is a $B_\lambda[A]$ set.

If $\gamma = Ch_N(\lambda)$, then 1. $f_\gamma \leq Ch_N$. 2. For every $h \in N \cap \Pi A$ there exists some $d \in N \cap \Pi A$ such that $$h \restriction B <_{J_{<\lambda}} d \restriction B \text{ and } d \leq f_\gamma,$$ where $B = B_\lambda[A]$.	(I.33)

We have seen that any persistently cofinal sequence for λ satisfies (I.32) (this is Lemma 5.4), and that any universal, minimally obedient sequence satisfies (I.33) as well (by Lemma 5.7).

Suppose that f is a sequence of length λ and $N \prec H_\Psi$ is κ-presentable and such that $f \in N$ (so $A, \lambda \in N$). Suppose that both (I.32) and (I.33) hold. If $\gamma = Ch_N(\lambda)$, then $f_\gamma \leq Ch_N$ by (I.33), and hence

$$\{a \in A \mid Ch_N(a) = f_\gamma(a)\} \tag{I.34}$$

is a $B_\lambda[A]$ set by (I.32). We shall use this observation in the following.

5.8 Lemma. *Suppose that A is a progressive set of regular cardinals and κ is a regular cardinal such that $|A| < \kappa < \min A$. Suppose that $\lambda_0 \in \text{pcf}(A)$ and $f^{\lambda_0} = \langle f^{\lambda_0}_\xi \mid \xi < \lambda_0 \rangle$ is a sequence of functions in ΠA. Let $N \prec H_\Psi$ be a κ-presentable elementary substructure (Ψ is a sufficiently large cardinal) such that $A, \lambda_0, f^{\lambda_0} \in N$. Suppose that N and $f^{\lambda_0} \in N$ satisfy properties (I.32) and (I.33) for $\lambda = \lambda_0$. Put $\gamma_0 = Ch_N(\lambda_0)$ and define*

$$b_{\lambda_0} = \{a \in A \mid Ch_N(a) = f^{\lambda_0}_{\gamma_0}(a)\}.$$

Then the following hold.

1. *b_{λ_0} is a $B_{\lambda_0}[A]$ set, namely*
$$J_{\leq\lambda_0}[A] = J_{<\lambda_0}[A] + b_{\lambda_0}.$$
2. *There exists a set $b'_{\lambda_0} \subseteq b_{\lambda_0}$ such that*
   (a) *$b'_{\lambda_0} \in N$*
   (b) *$b_{\lambda_0} \setminus b'_{\lambda_0} \in J_{<\lambda_0}[A]$ (hence b'_{λ_0} is also a B_{λ_0} set).*

Proof. Note that since $f^{\lambda_0}_{\gamma_0} \leq Ch_N$, $b_{\lambda_0} = \{a \in A \mid Ch_N(a) \leq f^{\lambda_0}_{\gamma_0}(a)\}$. We have already observed in the paragraph preceding the lemma that *1* holds.

We prove *2.* As the definition of b_{λ_0} involves N and $f^{\lambda_0}_{\gamma_0}$, we do not expect that $b_{\lambda_0} \in N$. However we shall find an inner approximation b'_{λ_0} of b_{λ_0} that lies in N. If $a \in A$ and $f^{\lambda_0}_{\gamma_0}(a) < Ch_N(a)$, then there exists some $\alpha < \kappa$ such that $f^{\lambda_0}_{\gamma_0}(a) < Ch_{N_\alpha}(a)$ (because $N = \bigcup_{\alpha<\kappa} N_\alpha$). Since $|A| < \kappa$, there is some sufficiently large $\alpha < \kappa$ such that

$$f^{\lambda_0}_{\gamma_0}(a) < Ch_N(a) \text{ iff } f^{\lambda_0}_{\gamma_0}(a) < Ch_{N_\alpha}(a)$$

holds for every $a \in A$. Or equivalently (by negating both sides)

$$a \in b_{\lambda_0} \text{ iff } Ch_{N_\alpha}(a) \le f^{\lambda_0}_{\gamma_0}(a).$$

That is we have replaced the parameter N with N_α in the definition of b_{λ_0}, but γ_0 is still a parameter not in N.

Since f^{λ_0} satisfies (I.33), there exists (for $h = Ch_{N_\alpha}$) some function $d \in N$ such that

1. $Ch_{N_\alpha} \restriction B_{\lambda_0} <_{J_{<\lambda_0}} d \restriction B_{\lambda_0}$ and
2. $d \le f^{\lambda_0}_{\gamma_0}$.

Define

$$b'_{\lambda_0} = \{a \in A \mid Ch_{N_\alpha}(a) \le d(a)\}.$$

Now all parameters are in N and clearly $b'_{\lambda_0} \in N$. Property 1 above implies that for almost all $a \in B_{\lambda_0}$, $Ch_{N_\alpha}(a) < d(a)$ (i.e. except on a $J_{<\lambda_0}$ set). Hence $B_{\lambda_0} \subseteq_{J_{<\lambda_0}} b'_{\lambda_0}$. Property 2 implies that $b'_{\lambda_0} \subseteq b_{\lambda_0}$. $\dashv$

Suppose that for every $\lambda \in \mathrm{pcf}(A)$ we attach a certain $B_\lambda[A]$ set b^*_λ. Then the Compactness Theorem (4.11) gives a finite set $\lambda_0, \dots, \lambda_{n-1}$ of $\mathrm{pcf}(A)$ cardinals such that $A = b^*_{\lambda_0} \cup \dots \cup b^*_{\lambda_{n-1}}$. Now let $N \prec H_\Psi$ be such that $A \in N$ and assume that the sets b^*_λ are chosen in N for each $\lambda \in \mathrm{pcf}(A) \cap N$. Then the covering cardinals $\lambda_0, \dots, \lambda_{n-1}$ can be found in N, even when the map $\lambda \mapsto b^*_\lambda$ is not in N. To prove that, we define a descending sequence of cardinals $\lambda_0 > \dots > \lambda_i$ by induction on i, starting with $\lambda_0 = \max\ \mathrm{pcf}(A)$, so that the following two conditions hold.

1. $\lambda_i \in N$.
2. If $A_k = A \setminus (b^*_0 \cup \dots \cup b^*_{k-1}) \ne \emptyset$, then $\lambda_k = \max\ \mathrm{pcf}(A_k)$.

Since $b^*_0, \dots, b^*_{k-1}$ are in N, $A_k \in N$ as well, and hence $\lambda_k \in N$ (whenever $A_k \ne \emptyset$ and λ_k is defined). It follows from lemmas 4.14 and 4.10 that $\lambda_0 > \lambda_1 > \dots > \lambda_k$. Hence, for some k, $A_k = \emptyset$, and then $A = b^*_0 \cup \dots \cup b^*_{k-1}$.

Here is a main result saying that the number of characteristic functions $Ch_N \restriction A$ is bounded by $\max\ \mathrm{pcf}(A)$.

5.9 Corollary. *Suppose that A is a progressive set of regular cardinals, κ is a regular cardinal such that $|A| < \kappa < \min(A)$, and N with $A \in N$ is a κ-presentable elementary substructure $N \prec H_\Psi$ and containing, for every $\lambda \in \mathrm{pcf}(A) \cap N$, a sequence $f^\lambda = \langle f^\lambda_\xi \mid \xi < \lambda \rangle$ that satisfies properties (I.32) and (I.33). Then there are cardinals $\lambda_0 > \lambda_1 \cdots > \lambda_n$ in $N \cap \mathrm{pcf}(A)$ such that*

$$Ch_N \restriction A = \sup\{f^{\lambda_0}_{\gamma_0}, \ldots, f^{\lambda_n}_{\gamma_n}\}, \tag{I.35}$$

where $\gamma_i = Ch_N(\lambda_i)$.

Proof. We employ Lemma 5.8, which assigns $B_\lambda[A]$ sets, $b'_\lambda \in N$, for every $\lambda \in \mathrm{pcf}(A) \cap N$, so that

$$b'_\lambda \subseteq \{a \in A \mid Ch_N(a) = f^\lambda_{Ch_N(\lambda)}(a)\}. \tag{I.36}$$

By the Inductive Covering procedure explained above, for some $\lambda_0, \ldots, \lambda_{n-1}$ in $\mathrm{pcf}(A) \cap N$

$$A = b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_{n-1}}.$$

Since property (I.33) ensures that $f^\lambda_{Ch_N(\lambda)} \le Ch_N$, (I.36) implies that (I.35) holds. ⊣

Application: the cofinality of $([\mu]^\kappa, \subseteq)$

For cardinals $\kappa \le \mu$, let $[\mu]^\kappa$ denote the collection of all subsets of μ of size κ. Under the inclusion relation $\subseteq$ this collection is a partial ordering, and we denote its cofinality by $\mathrm{cf}([\mu]^\kappa, \subseteq)$. Likewise, $[\mu]^{<\kappa}$ is the collection of all subsets of μ of cardinality less than κ. For example, if μ is a regular cardinal then the collection of all proper initial segments of μ is cofinal in $[\mu]^{<\mu}$.

One reason for the importance of studying $\mathrm{cf}([\mu]^\kappa, \subseteq)$ is the relationship

$$|[\mu]^\kappa| = \mathrm{cf}([\mu]^\kappa, \subseteq) \cdot 2^\kappa \tag{I.37}$$

and its applications to cardinal arithmetic (which we shall see). The proof of (I.37) is quite simple. Suppose that $\mathrm{cf}([\mu]^\kappa, \subseteq) = \lambda$ and let $Y = \{Y_i \in [\mu]^\kappa \mid i < \lambda\}$ be cofinal. A one-to-one map from $[\mu]^\kappa$ to $Y \times 2^\kappa$ can be defined as follows. For every $E \in [\mu]^\kappa$ find some $E \subseteq Y_i$. Since Y_i has cardinality κ, E is isomorphic to some subset S of κ, and then we map E to $\langle Y_i, S \rangle$.

We record some relatively simple facts about cofinalities of $[\mu]^\kappa$.

5.10 Lemma. *For any cardinal μ:*

1. *If $\kappa_1 < \kappa_2$ then*

$$\mathrm{cf}([\mu]^{\kappa_1}, \subseteq) \le \mathrm{cf}([\mu]^{\kappa_2}, \subseteq) \cdot \mathrm{cf}([\kappa_2]^{\kappa_1}, \subseteq).$$

2. *If* $\mu_1 < \mu_2$ *then* $\mathrm{cf}([\mu_1]^\kappa, \subseteq) \leq \mathrm{cf}([\mu_2]^\kappa, \subseteq)$.

3. *Suppose that* $\kappa \leq \mu$ *and* $E \subseteq [\mu]^\kappa$ *is cofinal. Then there exists a cofinal set in* $([\mu^+]^\kappa, \subseteq)$ *of cardinality* $|E| \cdot \mu^+$.

Proof. We prove the third item. For every $\mu \leq \gamma < \mu^+$ let f_γ be a bijection from γ to μ. Then the collection of all sets of the form $f_\gamma^{-1}\ X$, where $X \in E$, is cofinal and of cardinality $|E| \cdot \mu^+$. $\dashv$

A consequence (which can be proved by induction) is that for every $n < \omega$, $\mathrm{cf}([\aleph_n]^{\aleph_0}, \subseteq) = \aleph_n$.

The first application of the pcf theory to the subset cofinality question is the following

5.11 Theorem. *Suppose that* μ *is a singular cardinal, and* $\kappa < \mu$ *is an infinite cardinal such that the interval* A *of regular cardinals in* (κ, μ) *has size* $\leq \kappa$. *Then*

$$\mathrm{cf}([\mu]^\kappa, \subseteq) = \max\ \mathrm{pcf}(A).$$

Proof. Let μ and κ be as in the theorem. Define

$$A = \{\gamma \mid \gamma \text{ is a regular cardinal and } \kappa < \gamma < \mu\}.$$

We assume that $|A| \leq \kappa$, so that A is a progressive interval of regular cardinals. To prove the theorem, we first prove the easier inequality $\geq$. Let $\{X_i \mid i \in I\} \subseteq [\mu]^\kappa$ be cofinal and of cardinality $\mathrm{cf}([\mu]^\kappa, \subseteq)$. Define $h_i = Ch_{X_i} \restriction A$. That is, $h_i(a) = \sup\ a \cap X_i$ for $a \in A$. Then $\{h_i \mid i \in I\}$ is cofinal in $(\Pi A, <)$. (Since for every $f \in \Pi A$ the range of f is a subset of μ of size $\leq |A| \leq \kappa$, and is hence covered by some X_i. So $f \leq h_i$.) But we know that the cofinality of $(\Pi A, <)$ is $\max\ \mathrm{pcf}(A)$, and hence $|I| \geq \max\ \mathrm{pcf}(A)$.

Now we prove the $\leq$ inequality. We assume first that $|A| < \kappa$ and prove the $\leq$ inequality for this case. Then we can obtain the $|A| = \kappa$ case by applying the first case to κ^+ (instead of κ) and using

$$\mathrm{cf}([\mu]^\kappa, \subseteq) \leq \mathrm{cf}([\mu]^{\kappa^+}, \subseteq) \cdot \kappa^+.$$

So assume that $|A| < \kappa$ (and hence κ is uncountable). We plan to present a cofinal subset of $[\mu]^\kappa$ of cardinality $\max\ \mathrm{pcf}(A)$. Fix for every $\rho \in \mathrm{pcf}(A)$ a minimally obedient (at cofinality κ) universal sequence for ρ, and let $f = \{f^\rho \mid \rho \in \mathrm{pcf}(A)\}$ be the resulting array of sequences. In fact, we let f be the minimal such array in the well-ordering $<^*$ of H_Ψ, so that $f \in M$ for every $M \prec H_\Psi$ such that $A \in M$. Let $\mathcal{M}$ be the collection of all substructures $M \prec H_\Psi$ that are κ-presentable and such that $A \in M$ (so $A \subseteq M$). We know that (I.32) and (I.33) hold. Consider the collection $F = \{M \cap \mu \mid M \in \mathcal{M}\}$. This collection is clearly cofinal in $[\mu]^\kappa$, since for any set $X \in [\mu]^\kappa$ a structure $M \in \mathcal{M}$ can be defined so that $X \subseteq M$

(or even $X \in M$). We shall prove that $|F| \leq \max\ \mathrm{pcf}(A)$. We know (by Corollary 5.9) that for every $M \in \mathcal{M}$, $Ch_M \restriction A$ is the supremum of a finite number of functions taken from the array $\{f^\rho \mid \rho \in \mathrm{pcf}(A)\}$, which contains $\max\ \mathrm{pcf}(A)$ functions. Hence it suffices to prove that whenever $M, N \in \mathcal{M}$ are such that $Ch_M \restriction A = Ch_N \restriction A$, then $M \cap \mu = N \cap \mu$. But this is exactly Theorem 5.2. $\dashv$

The theorem just proved (5.11) has important consequences for cardinal arithmetic which we shall explore now. Look, for example, at $\mu = \aleph_\omega$, $\kappa = \aleph_0$, and $A = \{\aleph_n \mid 1 < n < \omega\}$. Then

$$\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \max\ \mathrm{pcf}(A).$$

So $\aleph_\omega^{\aleph_0} = (\max\ \mathrm{pcf}(A))2^{\aleph_0}$. If $\aleph_\omega$ is a strong limit cardinal then $[\aleph_\omega]^\omega$ has cardinality $2^{\aleph_\omega}$, and this cardinal turns out to be regular since it is $\max\ \mathrm{pcf}(A)$. Similarly, for every $n < \omega$, $\mathrm{cf}([\aleph_\omega, \subseteq]^{\aleph_n}, \subseteq) = \max\ \mathrm{pcf}(A)$. Hence

$$\mathrm{cf}([\aleph_\omega]^{\aleph_n}, \subseteq) = \mathrm{cf}([\aleph_\omega]^{\aleph_m}, \subseteq)$$

for every $n, m < \omega$.

Since A is an interval of regular cardinals, $\mathrm{pcf}(A)$ is also an interval of regular cardinals (Theorem 3.9) containing all regular cardinals from $\aleph_2$ to $\max\ \mathrm{pcf}(A)$. Hence if we write $\max \mathrm{pcf}(A) = \aleph_\alpha$, then $|\alpha| = |\mathrm{pcf}(A)|$ follows. Yet $|\mathrm{pcf}(A)| \leq 2^{\aleph_0}$ (Theorem 3.6). Thus $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \aleph_\alpha$ for $\alpha < (2^{\aleph_0})^+$. Thus we have proved the following theorem.

5.12 Theorem. $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{(2^{\aleph_0})^+}$.

An immediate conclusion is

5.13 Theorem. $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$.

Proof. If $2^{\aleph_0} > \aleph_\omega$ (equality is impossible by König's theorem) then $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$, and then $2^{\aleph_0} \leq \aleph_{2^{\aleph_0}}$ implies the theorem as a triviality. So we assume that $2^{\aleph_0} < \aleph_\omega$.

Suppose that $\aleph_\alpha = \mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$. We have proved in the preceding theorem that $\alpha < (2^{\aleph_0})^+$. Let $\{X_i \mid i < \aleph_\alpha\} \subseteq [\aleph_\omega]^{\aleph_0}$ be cofinal. So $[\aleph_\omega]^{\aleph_0} \subseteq \bigcup\{\mathcal{P}(X_i) \mid i < \aleph_\alpha\}$. Hence $|[\aleph_\omega]^{\aleph_0}| \leq 2^{\aleph_0} \cdot \aleph_\alpha = \aleph_\alpha < \aleph_{(2^{\aleph_0})^+}$. $\dashv$

We want to generalize this theorem to arbitrary singular cardinals $\aleph_\delta$ such that $\delta < \aleph_\delta$. A straightforward generalization gives the following which we leave as an exercise: If δ is a limit ordinal such that $\delta < \aleph_\delta$, then

$$\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{(2^{|\delta|})^+}$$

and hence

$$\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}.$$

We shall describe now a tighter bound: $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$.

As in the proof for bounding $\aleph_\omega^{\aleph_0}$, which consists in first evaluating the cofinality of $([\aleph_\omega]^{\aleph_0}, \subseteq)$, here too we first investigate cardinalities of covering sets. For cardinals $\mu \geq \tau$ a *cover* for $[\mu]^{<\tau}$ is a collection $\mathcal{C}$ of subsets of μ such that for every $X \in [\mu]^{<\tau}$ there exists $Y \in \mathcal{C}$ with $X \subseteq Y$. For cardinals $\mu \geq \theta \geq \tau$, $\mathrm{cov}(\mu, \theta, \tau)$ is the least cardinality of a cover for $[\mu]^{<\tau}$ consisting of sets taken from $[\mu]^{<\theta}$. So $\mathrm{cov}(\mu, \theta, \tau)$ measures how many sets, each of cardinality $< \theta$, are needed to cover every subset of μ of cardinality $< \tau$. For example, $\mathrm{cf}([\mu]^\kappa, \subseteq) = \mathrm{cov}(\mu, \kappa^+, \kappa^+)$. We shall prove the following.

5.14 Theorem. *Suppose that μ is a singular cardinal, and $\kappa < \mu$ a regular cardinal. Let A be the set of all regular cardinals in the interval $[\kappa^{++}, \mu)$. If $|A| \leq \kappa$, then*

$$\mathrm{cov}(\mu, \kappa^+, \mathrm{cf}(\mu)^+) = \sup\ \mathrm{pcf}_{\mathrm{cf}(\mu)}(A).$$

(See Definition 3.10 for $\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$.)

Before proving this theorem, let's see how it can be employed.

5.15 Corollary. *Suppose that δ is a limit ordinal such that $\delta < \aleph_\delta$. Then*

$$\mathrm{cov}(\aleph_\delta, |\delta|^+, \mathrm{cf}(\delta)^+) < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$$

and hence

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}.$$

Proof. Suppose that δ is a limit ordinal such that $\delta < \aleph_\delta$. Let $\mu = \aleph_\delta$, and $\kappa = |\delta|^+$. Define A as the set of all regular cardinals in the interval $(\kappa{+}{+}, \mu]$. So $|A| \leq |\delta|$. By Theorem 5.14, there exists a collection $\{X_i \mid i \in I\}$, where $X_i \in [\mu]^\kappa$ and $|I| = \sup\ \mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$, such that for every $Z \in [\aleph_\delta]^{\mathrm{cf}(\mu)}$, $Z \subseteq X_i$ for some $i \in I$. Yet, by Theorem 3.11, $\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$ is also an interval of regular cardinals, containing all regular cardinals in the interval $[\kappa^{++}, \aleph_\alpha)$ where $\aleph_\alpha = \sup\ \mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$. Now $|\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)| \leq |[A]^{\mathrm{cf}(\mu)}| \cdot 2^{\mathrm{cf}(\mu)} \leq |\delta|^{\mathrm{cf}(\mu)}$. It follows (see the proof in the following paragraph) that $\alpha < (|\delta|^{\mathrm{cf}(\mu)})^+$. That is, $|I| < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$ (as $\mathrm{cf}(\mu) = \mathrm{cf}(\delta)$). Hence $|[\aleph_\delta]^{\mathrm{cf}(\delta)}| < \kappa^{\mathrm{cf}(\delta)} \cdot \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$. Thus $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$ as required.

We prove that $\alpha < (|\delta|^{\mathrm{cf}(\mu)})^+$. Since $\delta < |\delta|^+ \leq (|\delta|^{\mathrm{cf}(\mu)})^+$, it follows that the interval $(\aleph_\delta, \aleph_{(|\delta|^{\mathrm{cf}(\mu)})^+})$ contains $(|\delta|^{\mathrm{cf}(\mu)})^+$ regular cardinals. But the interval of regular cardinals in $(\aleph_\delta, \aleph_\alpha)$ is included in $\mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$ and contains $\leq |\delta|^{\mathrm{cf}(\mu)}$ regular cardinals. Hence $\alpha < (|\delta|^{\mathrm{cf}(\mu)})^+$. $\dashv$

We proceed now with the *proof of Theorem 5.14*. Let $\kappa < \mu$ and $|A| \leq \kappa$ be as in the theorem. Since A is cofinal in μ and $|A| \leq \kappa$, $\mathrm{cf}(\mu) \leq \kappa$. Let $\rho = \mathrm{cf}(\mu)$ be the cofinality of μ. We shall prove that $\mathrm{cov}(\mu, \kappa^+, \rho^+) = \sup\ \mathrm{pcf}_\rho(A)$.

For the $\geq$ inequality, we must prove that $\mathrm{cov}(\mu, \kappa^+, \rho^+) \geq \lambda$ for every $\lambda \in \mathrm{pcf}_\rho(A)$. That is, if $A_0 \subseteq A$ is of cardinality ρ we want to prove that $\mathrm{cov}(\mu, \kappa^+, \rho^+) \geq \max\ \mathrm{pcf}(A_0)$. So let $\{X_i \mid i \in I\}$ be a covering of $[\mu]^{\mathrm{cf}(\mu)}$ with sets X_i of cardinality $\leq \kappa$. For each X_i define $h_i = Ch_{X_i} \restriction A_0$. Then $\{h_i \mid i \in I\}$ is cofinal in $(\Pi A_0, <)$, and hence $|I| \geq \max\ \mathrm{pcf}(A_0)$.

For the $\leq$ inequality, we must provide a covering set for $\mathrm{cov}(\mu, \kappa^+, \rho^+)$ of cardinality $\sup\ \mathrm{pcf}_\rho(A)$.

For every $\lambda \in \mathrm{pcf}_\rho(A)$, $\lambda \in \mathrm{pcf}(A)$ as well, and we fix a minimally obedient at cofinality ρ^+ sequence $f^\lambda = \langle f^\lambda_\xi \mid \xi < \lambda \rangle$ of functions in ΠA that is universal for λ.

For every $\alpha < \mu$ such that $\mathrm{cf}(\alpha) = \rho^+$, let $E_\alpha \subseteq \alpha$ be a closed unbounded subset of α of order type ρ^+.

Define $\mathcal{F}$ as the collection of all functions of the form $\sup\{f^{\lambda_1}_{\alpha_1}, \cdots, f^{\lambda_n}_{\alpha_n}\}$ where $\lambda_i \in \mathrm{pcf}_\rho(A)$ and $\alpha_i < \lambda_i$. Clearly $\mathcal{F}$ has cardinality $\sup\ \mathrm{pcf}_\rho(A)$. For every $f \in \mathcal{F}$ let

$$E(f) = \bigcup \{E_{f(a)} \mid a \in A \text{ and } \mathrm{cf}(f(a)) = \rho^+\}$$

Then the cardinality of $E(f)$ is at most κ^+. Let

$$K(f) = \mathrm{Skolem}(E(f) \cup \kappa^+) \prec H_\Psi$$

be the Skolem hull (closure) of $E(f) \cup \kappa^+$. We remind the reader that the structure H_Ψ includes a class well-ordering $<^*$ of all sets, and hence there is a countable set of Skolem functions for H_Ψ so that $X \prec H_\Psi$ iff X is closed under all of these Skolem functions. The cardinality of $K(f)$ is κ^+.

Clearly $\mathcal{K} = \{K(f) \mid f \in \mathcal{F}\}$ has cardinality $\leq \sup\ \mathrm{pcf}_\rho(A)$. Our aim now is to show that

$$\mathcal{K} \text{ covers } [\mu]^{\mathrm{cf}(\mu)}.$$

Since $\mathrm{cf}([\kappa^+]^\kappa, \subseteq) = \kappa^+$, this yields that

$$\mathrm{cov}(\mu, \kappa^+, \mathrm{cf}(\mu)^+) = \sup\ \mathrm{pcf}_{\mathrm{cf}(\mu)}(A).$$

Let $Z \subset \mu$ be of size $\rho = \mathrm{cf}(\mu)$. Define $\langle M_i \mid i < \rho^+ \rangle$ an increasing and continuous chain of elementary substructures $M_i \prec H_\Psi$, each of cardinality ρ, such that $A, Z \in M_0$, $M_i \in M_{i+1}$, and $Z \subset M_0$. Let $M = \bigcup_{i<\rho^+} M_i$ be the resulting ρ^+-presentable structure.

For every $a \in A \cap M$ (and in fact for every $a \in A$), $Ch_M(a)$ has cofinality ρ^+. Indeed $\langle Ch_{M_i}(a) \mid i < \rho^+ \rangle$ is increasing, continuous and with limit $Ch_M(a)$. There is another closed unbounded sequence in $Ch_M(a)$ which interests us, namely $E_{Ch_M(a)}$, and we consider the intersection of these two closed unbounded sets. So there exists a closed unbounded set $D_a \subseteq \rho^+$ such that for every $i \in D_a$

$$Ch_{M_i}(a) \in E_{Ch_M(a)}. \tag{I.38}$$

For every $i < \rho^+$, M_i has cardinality ρ and hence $D(i) = \bigcap\{D_a \mid a \in A \cap M_i\}$ is closed unbounded in ρ^+. Form the diagonal intersection $D = \{j \in \rho^+ \mid \forall i < j\ (j \in D(i))\}$. Fix any $j_0 \in D'$ (a limit point of D). For every $a \in A \cap M_{j_0}$ there exists some $j_1 < j_0$ such that $a \in A \cap M_{j_1}$. If $j_1 < i \le j_0$ and $i \in D$, then $i \in D(j_1)$ and hence $i \in D_a$. So $Ch_{M_i}(a) \in E_{Ch_M(a)}$. Thus $\langle Ch_{M_i}(a) \mid j_1 < i < j_0 \wedge i \in D\rangle$ is a sequence of ordinals in $M_{j_0} \cap E_{Ch_M(a)}$ that tends to $Ch_{M_{j_0}}(a)$ (whenever $j_0 \in D'$ and $a \in A \cap M_{j_0}$).

Define $A_0 = A \cap M_{j_0}$. Then $A_0 \in [A]^{\mathrm{cf}(\mu)}$, and $A_0 \in M$. We plan to apply Corollary 5.9 to A_0, ρ^+ and M (substituting A, κ, and N there). For every $\lambda \in \mathrm{pcf}(A_0)$, $\lambda \in \mathrm{pcf}_{\mathrm{cf}(\mu)}(A)$ and the sequence $\langle f^\lambda_\xi \restriction A_0 \mid \xi < \lambda\rangle$ is, in M, universal for λ and minimally obedient at ρ^+. Hence, by 5.9,

$$Ch_M \restriction A_0 = f \restriction A_0 \text{ for some } f \in \mathcal{F}. \tag{I.39}$$

Since $Z \subseteq M_{j_0}$, the following proves that $Z \subseteq K(f)$.

Claim. $M_{j_0} \cap \mu \subseteq K(f)$.

By Lemma 5.2, this is a consequence of the following

5.16 Lemma. *For every successor cardinal $\sigma^+ \in M_{j_0} \cap \mu$*

$$\sup\,(M_{j_0} \cap \sigma^+) = \sup\,(M_{j_0} \cap K(f) \cap \sigma^+).$$

Proof. Assume that $\sigma^+ \in M_{j_0} \cap \mu$. If $\sigma^+ \le \kappa^+$, then $\kappa^+ \subseteq K(f)$ implies the lemma immediately. So assume that $\sigma^+ > \kappa^+$, and hence that $\sigma^+ \in A \cap M_{j_0} = A_0$. Now (I.39) implies that $Ch_M(\sigma^+) = f(\sigma^+) = \alpha$. Hence $\mathrm{cf}(\alpha) = \rho^+$ and $E_\alpha \subseteq E(f) \subseteq K(f)$. The sequence $\langle Ch_{M_i}(\sigma^+) \mid j_1 < i < j_0 \wedge i \in D\rangle$ is unbounded in $Ch_{M_{j_0}}(\sigma^+)$, as we have observed above, and thus shows that the lemma is correct. $\dashv$

5.17 Exercise. 1. Let μ, κ, and A be as in Theorem 5.14. Suppose that $|A| \le \kappa$. Prove that

$$\mathrm{cov}(\mu, \kappa^+, \aleph_1) = \sup\ \mathrm{pcf}_{\aleph_0}(A).$$

Conclude that if $\delta < \aleph_\delta$ is a limit ordinal, then

$$\aleph_\delta^{\aleph_0} < \aleph_{(|\delta|^{\aleph_0})^+}.$$

Hint. By induction on μ.

2. Suppose that δ is a limit ordinal such that for every cardinal $\mu < \delta$ $\mu^{\aleph_0} < \delta$. Then $\aleph_\delta$ satisfies the same property, namely for every $\mu < \aleph_\delta$, $\mu^{\aleph_0} < \aleph_\delta$.

   Hint. Without loss of generality, $\delta < \aleph_\delta$. Prove that $\mu^{\aleph_0} < \aleph_\delta$ by induction on $\mu < \aleph_\delta$.

6. Elevations and transitive generators

Given a progressive set A of regular cardinals, we have proved the existence of generating sets $B_\lambda = B_\lambda[A]$. Suppose that N is such that $A \subseteq N \subseteq \mathrm{pcf}(A)$ and $B = \langle B_\lambda \mid \lambda \in N\rangle$ is a generating sequence (defined only for λ in N). Then B is said to be *smooth* (or transitive) if for every $\lambda \in N$ and $\theta \in B_\lambda$, $B_\theta \subseteq B_\lambda$.

This definition is trivial when $B_\theta = \{\theta\}$ (that is when $\theta \notin \mathrm{pcf}(A \cap \theta)$). However, we shall be interested in A's for which $\theta \in \mathrm{pcf}(A \cap \theta)$ is possible for $\theta \in A$. The reason for considering subsets N of $\mathrm{pcf}(A)$ in this definition, rather than the whole $\mathrm{pcf}(A)$ (which would be most desirable) is that we only know how to prove the existence of smooth sequences for sets N of cardinality $\min(A)$.

Our aim is to obtain transitive generators; they will be useful in proving, for example, that for every progressive interval of regular cardinals A, $|\mathrm{pcf}(A)| < |A|^{+4}$. However, there is still some material to cover beforehand.

Fix a progressive set A of regular cardinals and let κ be a regular cardinal such that $|A| < \kappa < \min(A)$. For every $\lambda \in \mathrm{pcf}(A)$ let $f^\lambda = \langle f^\lambda_\xi \mid \xi < \lambda\rangle$ be a universal sequence for λ which is minimally obedient (at cofinality κ). It is convenient to assume that for $a \in A \setminus \lambda$, $f^\lambda_\xi(a) = \xi$. The *elevation* of the array $\langle f^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ is another array $\langle F^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ of persistently cofinal sequences defined below, and which will be shown to satisfy properties (I.32) and (I.33).

For every finite, decreasing sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$ of cardinals such that $\lambda_0 \in \mathrm{pcf}(A)$ and $\lambda_{i+1} \in A \cap \lambda_i$ for $i < n$, and for every ordinal $\gamma_0 \in \lambda_0$, define a sequence $\gamma_1 \in \lambda_1, \ldots, \gamma_n \in \lambda_n$ by

$$\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1}). \tag{I.40}$$

So $\gamma_1 = f^{\lambda_0}_{\gamma_0}(\lambda_1)$, $\gamma_2 = f^{\lambda_1}_{\gamma_1}(\lambda_2)$, etc. until $\gamma_n = f^{\lambda_{n-1}}_{\gamma_{n-1}}(\lambda_n)$. Now define the elevation function $El_{\lambda_0,\ldots,\lambda_n}$ on λ_0 by

$$El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) = \gamma_n.$$

We say that the last value obtained, γ_n, is reached from $f^{\lambda_0}_{\gamma_0}$ via the descending sequence $\lambda_0 > \lambda_1 > \cdots > \lambda_n$.

Fix a cardinal $\lambda_0 \in \mathrm{pcf}(A)$. We want to define the elevated sequence F^{λ_0}, first on $A \cap \lambda_0$. Given any $\lambda \in A \cap \lambda_0$, let $F_{\lambda_0,\lambda}$ be the set of all finite, descending sequences $\langle \lambda_0 > \lambda_1 > \cdots > \lambda_n\rangle$ leading from λ_0 to $\lambda_n = \lambda$, such that λ_i for $i > 0$ are all in A. For every $\gamma_0 \in \lambda_0$ we ask whether there is a maximal value among

$$\{El_{\lambda_0,\ldots,\lambda_n}(\gamma_0) \mid \langle \lambda_0, \ldots, \lambda_n\rangle \in F_{\lambda_0,\lambda}\}.$$

If this set contains a maximum, let $F^{\lambda_0}_{\gamma_0}(\lambda)$ be that maximum, and otherwise put $F^{\lambda_0}_{\gamma_0}(\lambda) = f^{\lambda_0}_{\gamma_0}(\lambda)$. In case $\lambda \in A$ and $\lambda \geq \lambda_0$, define $F^{\lambda_0}_{\gamma_0}(\lambda) = \gamma_0$. So $F^{\lambda_0} = \langle F^{\lambda_0}_\gamma \mid \gamma < \lambda_0\rangle$ with $F^{\lambda_0}_\gamma \in \Pi A$ is defined.

The elevated array $\langle F^{\lambda_0} \mid \lambda_0 \in \mathrm{pcf}(A)\rangle$ thus defined will give the required transitive generating sequence. Observe first that

$$f^{\lambda_0}_\gamma \le F^{\lambda_0}_\gamma \text{ for every } \gamma < \lambda_0.$$

This is so because $El_{\lambda_0,\lambda}(\gamma_0) = f^{\lambda_0}_{\gamma_0}(\lambda)$ for every $\lambda \in A \cap \lambda_0$; so that this original value is among the values considered for maximum. Hence

$$F^{\lambda_0} \text{ is persistently cofinal for } \lambda_0.$$

This shows that Lemma 5.4 can be applied and property (I.32) holds whenever $F^\lambda \in N$ (and N is κ-presentable).

Another observation concerns any κ-presentable elementary substructure $N \prec H_\Psi$ such that $A, \langle f^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle \in N$. Being definable, the elevated array is also in N. Even though each f^λ is assumed to be minimally obedient, the elevated sequence F^λ is not anymore club-obedient. We have however the following consequence of Lemma 5.7.

6.1 Lemma. *If $\lambda_0 \in \mathrm{pcf}(A) \cap N$ and $\gamma_0 = Ch_N(\lambda_0)$, then for every $\lambda \in A \cap \lambda_0$, $F^{\lambda_0}_{\gamma_0}(\lambda) \in \bar{N} \cap \lambda$ (where $\bar{N}$ is the ordinal closure of N). Thus the elevated sequence F^{λ_0} satisfies (I.33). Namely,*

1. *$F^{\lambda_0}_{\gamma_0}(\lambda) \le Ch_N(\lambda)$ for every $\lambda \in A$, and*
2. *for every $h \in N \cap \Pi A$ there exists some $d \in N \cap \Pi A$ such that*

$$h \restriction B <_{J_{<\lambda_0}} d \restriction B \text{ and } d \le F^{\lambda_0}_{\gamma_0}$$

where $B = B_{\lambda_0}[A]$.

Proof. Observe first that $A \subseteq N$, $\lambda \in N$, and $F_{\lambda_0,\lambda} \subseteq N$. Consider any $\langle \lambda_0, \dots, \lambda_n\rangle \in F_{\lambda_0,\lambda}$ and the ordinals γ_i defined by (I.40). It follows from Lemma 5.7 that $\gamma_i \in \bar{N}$. If $\gamma_i \in N$ then obviously $\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1}) \in N$. If, however, $\gamma_i \in \bar{N} \setminus N$, then Lemma 5.7 yields that $f^{\lambda_i}_{\gamma_i}(a) \in \bar{N}$ for every $a \in A$, and in particular $\gamma_{i+1} \in \bar{N}$. Thus $El_{\lambda_0,\dots,\lambda_n}(\gamma_0) \in \bar{N}$ and hence $F^{\lambda_0}_{\gamma_0}(\lambda) \in \bar{N} \cap \lambda$.

Thus (I.33)(1) holds for F^{λ_0}. Since $f^{\lambda_0} \le F^{\lambda_0}$, where f^{λ_0} is universal and minimally obedient at κ, (I.33)(2) holds as well. ⊣

6.2 Lemma. *Let A, f, and N be as in the previous lemma. Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$, $\gamma_0 = Ch_N(\lambda_0)$ and $\lambda \in A \cap \lambda_0$.*

1. *If for some descending sequence $\lambda_0 > \cdots > \lambda_n = \lambda$ in $F_{\lambda_0,\lambda}$*

$$El_{\lambda_0,\dots,\lambda_n}(\gamma_0) = Ch_N(\lambda).$$

Then $Ch_N(\lambda)$ is the maximal value in $\{El_{\bar\lambda}(\gamma_0) \mid \bar\lambda \in F_{\lambda_0,\lambda}\}$ and hence

$$Ch_N(\lambda) = F^{\lambda_0}_{\gamma_0}(\lambda).$$

2. *Suppose that*
$$F^{\lambda_0}_{\gamma_0}(\lambda) = \gamma.$$
For any $a \in A \cap \lambda$, if
$$F^{\lambda}_{\gamma}(a) = Ch_N(a),$$
then
$$F^{\lambda_0}_{\gamma_0}(a) = Ch_N(a)$$
as well.

Proof. Item 1 says that if some descending sequence leading from λ_0 to λ reaches $Ch_N(\lambda)$, then no sequence reaches a higher value. But this is clear from Lemma 6.1 since $Ch_N(\lambda)$ is the maximal possible value.

Item 2 uses Item 1. It says that if γ can be reached from $f^{\lambda_0}_{\gamma_0}$ by a finite descending sequence leading to λ, and if there is another sequence leading from λ to a, so that $Ch_N(a)$ can be reached from f^{λ}_{γ}, then $Ch_N(a)$ can be reached already from $f^{\lambda_0}_{\gamma_0}$ via the concatenation of these descending sequences (and no higher value can be reached—by 1). $\dashv$

Now we can get our transitive generating sequence.

6.3 Theorem (Transitive Generators.). *Suppose that A is a progressive set of regular cardinals, and $|A| < \kappa < \min(A)$ is a regular cardinal. Let $\langle f^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ be an array of minimally obedient (at cofinality κ) universal sequences. Let $N \prec H_\Psi$ be an elementary substructure that is κ-presentable and such that A, $\langle f^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle \in N$. Let $\langle F^\lambda \mid \lambda \in \mathrm{pcf}(A)\rangle$ be the derived elevated array. For every $\lambda_0 \in \mathrm{pcf}(A) \cap N$ put $\gamma_0 = Ch_N(\lambda_0)$ and define*
$$b_{\lambda_0} = \{a \in A \mid Ch_N(a) = F^{\lambda_0}_{\gamma_0}(a)\}.$$
then the following hold:

1. *Every b_{λ_0} is a $B_{\lambda_0}[A]$ set, namely*
$$J_{\leq\lambda_0}[A] = J_{<\lambda_0}[A] + b_{\lambda_0}.$$

2. *There exists sets $b'_{\lambda_0} \subseteq b_{\lambda_0}$, for $\lambda_0 \in \mathrm{pcf}(A) \cap N$, such that*
   (a) *$b_{\lambda_0} \setminus b'_{\lambda_0} \in J_{<\lambda_0}[A]$.*
   (b) *$b'_{\lambda_0} \in N$ (but the sequence $\langle b'_{\lambda_0} \mid \lambda_0 \in \mathrm{pcf}(A) \cap N\rangle$ is not claimed to be in N).*

3. *The collection $\langle b_\lambda \mid \lambda \in \mathrm{pcf}(A) \cap N\rangle$ is transitive; which means that if $\lambda_1 \in b_\lambda$ then $b_{\lambda_1} \subseteq b_\lambda$.*

Proof. The elevated sequence F^{λ_0}satisfies properties (I.32) (because it is persistently cofinal) and (I.33) (as shown in Lemma 6.1). Thus items *1* and *2* of our lemma follow from Lemma 5.8. Observe that $b_{\lambda_0} \subseteq \lambda_0 + 1$, since $B_{\lambda_0} \in J_{<\lambda_0^+}$.

Transitivity (item *3*) relies on Lemma 6.2. Suppose that $\lambda_0 \in \mathrm{pcf}(A) \cap N$ and $\lambda_1 \in b_{\lambda_0}$. This means

$$Ch_N(\lambda_1) = F^{\lambda_0}_{\gamma_0}(\lambda_1)$$

where $\gamma_0 = Ch_N(\lambda_0)$. Say $Ch_N(\lambda_1) = \gamma_1$. We have to show that $b_{\lambda_1} \subset b_{\lambda_0}$ in this case. So assume that $a \in b_{\lambda_1}$. This means

$$Ch_N(a) = F^{\lambda_1}_{\gamma_1}(a).$$

Now Lemma 6.2(*2*) applies and yields

$$F^{\lambda_0}_{\gamma_0}(a) = Ch_N(a)$$

which gives $a \in b_{\lambda_0}$. ⊣

Localization

Localization is the following property of the pcf function which will be proved in this subsection.

> If A is a progressive set of regular cardinals and $B \subseteq \mathrm{pcf}(A)$ is also progressive, then for every $\lambda \in \mathrm{pcf}(B)$ there exists $B_0 \subseteq B$ such that $|B_0| \leq |A|$ and $\lambda \in \mathrm{pcf}(B_0)$.

The localization property implies that there exists no $B \subseteq \mathrm{pcf}(A)$ with $|B| = |A|^+$ and such that $b > \max\ \mathrm{pcf}(B \cap b)$ for every $b \in B$. For indeed if there were such B it would be progressive, and if we define $\lambda = \max\ \mathrm{pcf}(B)$, then λ is not in the pcf of any proper initial segment of B. In fact, $\lambda > \max\ \mathrm{pcf}(B_0)$ for any proper initial segment B_0 of B. It is this conclusion, the simplest case of localization, which is proved first.

6.4 Theorem. *Assume that A is a progressive set of regular cardinals. Then there is no set $B \subseteq \mathrm{pcf}(A)$ such that $|B| = |A|^+$, and, for every $b \in B$, $b > \max\ \mathrm{pcf}(B \cap b)$.*

Proof. Assume on the contrary that A is as in the theorem and yet, for some $B \subseteq \mathrm{pcf}(A)$ of cardinality $|A|^+$, $b > \max\ \mathrm{pcf}(B \cap b)$ for every $b \in B$. Since A is progressive $|A| < \min\ A$, and in case $|A|^+ \in A$ we may remove the first cardinal of A and assume that $|A|^+ < \min\ A$. The set $E = A \cup B$ of cardinality $|A|^+$ thus satisfies $|E| < \min\ E$ and the Transitive Generators Theorem 6.3 can be applied to E.

Find a κ-presentable elementary substructure, $N \prec H_\Psi$, that contains A and B where $\kappa = |E|$. Let $\langle b_\lambda \mid \lambda \in \mathrm{pcf}(E) \cap N\rangle$ be the set of transitive generators (subsets of E) as guaranteed by Theorem 6.3. Let $b'_\lambda \in N$ be such that $b'_\lambda \subseteq b_\lambda$ and $b_\lambda \setminus b'_\lambda \in J_{<\lambda}$.

Since $|A| < |B|$ we can find an initial segment $B_0 \subseteq B$ of cardinality $|A|$ such that if an arbitrary $a \in A$ is in some b_β, $\beta \in B$, then it is already in some b_β with $\beta \in B_0$. Namely

$$\forall a \in A\ [\ (\exists \beta \in B)\ a \in b_\beta \implies (\exists \beta \in B_0)\ a \in b_\beta\ ]. \tag{I.41}$$

Let $\beta_0 = \min(B \setminus B_0)$. So $B_0 = B \cap \beta_0$ and $B_0 \in N$.

Claim. *There exists a finite descending sequence of cardinals $\lambda_0 > \cdots > \lambda_n$ in $N \cap \mathrm{pcf}(B_0)$ such that*

$$B_0 \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}. \tag{I.42}$$

Proof. In fact we shall find a finite sequence $\lambda_0, \ldots, \lambda_n \in N \cap \mathrm{pcf}(B_0)$ such that $B_0 \subseteq b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_n}$. The proof is the same as that of Theorem 4.11, but one must be a little bit more careful to ensure that the pcf index-cardinals are in N.

So let $\lambda_0 = \max\ \mathrm{pcf}(B_0)$. Clearly $\lambda_0 \in N$ and hence $b'_{\lambda_0} \in N$. So $B_1 = B_0 \setminus b'_{\lambda_0} \in N$, and $\lambda_1 = \max\ \mathrm{pcf}(B_1) \in N \cap \lambda_0$. Next define $B_2 = B_1 \setminus b'_{\lambda_1}$ etc. The point is that we have $B_i \in N$ since $b'_{\lambda_{i-1}} \in N$, and we must stop with $B_{n+1} = \emptyset$ after a finite number of steps since $\lambda_0 > \lambda_1 \ldots$. Since $b'_{\lambda_i} \subseteq b_{\lambda_i}$, (I.42) holds. ⊣

The following claim will bring the desired contradiction and thus prove the theorem. Recall that $\beta_0 = \min(B \setminus B_0)$ and thus $\beta_0 > \max(\mathrm{pcf}(B_0)) \geq \lambda_0, \ldots, \lambda_n$. Since $\beta_0 \in \mathrm{pcf}(A)$, $\beta_0 \in \mathrm{pcf}(b_{\beta_0} \cap A)$ (or else $\beta_0 \in \mathrm{pcf}(A \setminus b_{\beta_0})$ which is impossible by Lemma 4.14). Yet the following inclusion shows that this is impossible.

6.5 Claim. $b_{\beta_0} \cap A \subseteq b_{\lambda_0} \cup \ldots \cup b_{\lambda_n}$.

Proof. Consider any cardinal $a \in b_{\beta_0} \cap A$. Then

$$a \in b_\beta$$

for some $\beta \in B_0$ (by I.41). As $B_0 \subseteq b_{\lambda_0} \cup \cdots \cup b_{\lambda_n}$, $\beta \in b_{\lambda_i}$ for some $0 \leq i \leq n$. But transitivity implies

$$b_\beta \subseteq b_{\lambda_i}$$

and hence

$$a \in b_{\lambda_i}$$

as required. This claim shows that $\max\ \mathrm{pcf}(b_{\beta_0} \cap A) < \beta_0$, and yet $\beta_0 \in \mathrm{pcf}(b_{\beta_0} \cap A)$ which is a contradiction! ⊣

Thus Theorem 6.4 is proved. ⊣

Now we pass to the general case and prove the localization theorem.

6.6 Theorem (Localization). *Suppose that A is a progressive set of regular cardinals. If $B \subseteq \mathrm{pcf}(A)$ is also progressive, then for every $\lambda \in \mathrm{pcf}(B)$ there exists $B_0 \subseteq B$ with $|B_0| \leq |A|$ and such that $\lambda \in \mathrm{pcf}(B_0)$.*

Proof. We prove by induction on λ that for every A and B as in the theorem the conclusion holds for λ. Replacing B with $B_\lambda[B]$, we may assume that $\lambda = \max\ \mathrm{pcf}(B)$.

6.7 Claim. *We may assume that the set $\lambda \cap \mathrm{pcf}(B)$ has no maximal cardinal.*

Proof. Suppose on the contrary the existence of some $\lambda_0 = \max\ (\lambda \cap \mathrm{pcf}(B))$. It is easy to remove λ_0 by defining

$$B_1 = B \setminus B_{\lambda_0}[B].$$

Then $\lambda \in \mathrm{pcf}(B_1)$ still holds since $B_{\lambda_0} \in J_{<\lambda}$. We can now replace B with B_1, and repeat, if necessary, this procedure a finite number of times until the claim holds (for some B_k which is renamed B). ⊣

We shall find now a set $C \subseteq \lambda \cap \mathrm{pcf}(B)$ of cardinality $\leq |A|$ such that $\lambda \in \mathrm{pcf}(C)$. Such C is necessarily progressive. Together with the inductive hypothesis this will conclude the proof; because for every $\gamma \in C$ we can pick $B(\gamma) \subseteq B$ of cardinality $\leq |A|$ and such that $\gamma \in \mathrm{pcf}\ B(\gamma)$, and then define $B_0 = \bigcup_{\gamma \in C} B(\gamma)$. Since $C \subseteq \mathrm{pcf}(B_0)$, $\lambda \in \mathrm{pcf}(B_0)$ will then follow from $\lambda \in \mathrm{pcf}(C)$ (by Theorem 3.12). So the following is the last piece of the proof.

6.8 Claim. *There exists a set $C \subseteq \lambda \cap \mathrm{pcf}(B)$ of cardinality $\leq |A|$ and such that $\lambda \in \mathrm{pcf}(C)$.*

Proof. Assume no such C exists. We shall construct a sequence $\langle \gamma_i \mid i \in |A|^+ \rangle$ of cardinals in $\mathrm{pcf}(B)$ such that

$$\gamma_i > \max\ \mathrm{pcf}\ \{\gamma_j \mid j < i\}.$$

This will contradict Theorem 6.4.

So suppose that $C = \{\gamma_j \mid j < i\}$ have been defined. Then

$$\lambda > \max\ \mathrm{pcf}(C).$$

Indeed $\lambda = \max\ \mathrm{pcf}(C)$ is impossible by our assumption that no such C exists, and $\lambda < \max\ \mathrm{pcf}(C)$ is impossible since $\mathrm{pcf}(C) \subseteq \mathrm{pcf}(B)$ and $\lambda = \max\ \mathrm{pcf}(B)$. We can find now $\gamma_i \in \mathrm{pcf}(B)$ above $\max\ \mathrm{pcf}(C)$ (recall that $\mathrm{pcf}(B)$ has no maximum below λ). ⊣

⊣

7. Size limitation on pcf of intervals

This relatively short section is devoted to a theorem which occupies a central place in the pcf theory and to a famous application:

$$\aleph_\omega^{\aleph_0} < \max\{(2^{\aleph_0})^+, \aleph_{\omega_4}\}.$$

The reader will notice that many of the ingredients developed so far appear in its proof. We know that for any A progressive set of regular cardinals the cardinality of $\mathrm{pcf}(A)$ does not exceed $2^{|A|}$, and it is an open question whether $|\mathrm{pcf}(A)| \leq |A|$ or not. At present the following theorem with its enigmatic appearance of the number four is the best result.

7.1 Theorem. *Let A be an interval of regular cardinals such that $|A| <$ $\min A$. Then*

$$|\mathrm{pcf}\ A| < |A|^{+4}.$$

Proof. Suppose that A is as in the theorem a progressive interval of regular cardinals, but $|\mathrm{pcf}(A)| \geq |A|^{+4}$. Say $|A| = \rho$. The following proof provides a sequence B of length ρ^+ of cardinals in $\mathrm{pcf}(A)$ such that each cardinal $b \in B$ is above $\max\ \mathrm{pcf}(B \cap b)$. This, of course, will be in contradiction to Theorem 6.4.

Let $S = S^{\rho^{+3}}_{\rho^+}$ be the set of ordinals in ρ^{+3} that have cofinality ρ^+. Choose a club guessing sequence $\langle C_k \mid k \in S\rangle$. So for every closed unbounded set $E \subseteq \rho^{+3}$ there exists some $k \in S$ such that $C_k \subset E$.

Consider the cardinal $\sup(A)$, and let σ be that ordinal such that $\aleph_\sigma = \sup(A)$. Since $\mathrm{pcf}(A)$ is an interval of regular cardinals (by Theorem 3.9), and since we assume that $\mathrm{pcf}(A)$ has cardinality at least ρ^{+4}, any regular cardinal in $\{\aleph_{\sigma+\alpha} \mid \alpha < \rho^{+4}\}$ is in $\mathrm{pcf}(A)$.

We intend to define a closed set $D \subset \rho^{+4}$ of order-type ρ^{+3}, $D = \{\alpha_i \mid i < \rho^{+3}\}$, and the impossible sequence of length ρ^+, B, will be a subset of $\{\aleph^+_{\sigma+\alpha} \mid \alpha \in D\}$. The definition of the ordinal α_i is by induction on $i < \rho^{+3}$.

1. For $i = 0$, $\alpha_0 = 0$.
2. If $i < \rho^{+3}$ is a limit ordinal, then $\alpha_i = \sup\{\alpha_j \mid j < i\}$.
3. Suppose that $\{\alpha_j \mid j \leq i\}$ has been defined for some $i < \rho^{+3}$, and we shall define α_{i+1}. Consider $i + 1 \subset \rho^{+3}$ as an isomorphic copy of $\{\alpha_j \mid j \leq i\}$. For every $k \in S$ look at the set $C_k \cap (i + 1)$ and define the set of cardinals $e_k = \{\aleph_{\sigma+\alpha_j} \mid j \in C_k \cap (i + 1)\}$. Then the set of successors $e_k^{(+)} = \{\gamma^+ \mid \gamma \in e_k\}$ is a set of regular cardinals, and we ask whether $\max\ \mathrm{pcf}(e_k^{(+)}) < \aleph_{\sigma+\rho^{+4}}$ or not. There are ρ^{+3} such questions, and therefore we can define $\alpha_{i+1} < \rho^{+4}$ so that $\alpha_i < \alpha_{i+1}$ and the following holds. For every $k \in S$, if $\max\ \mathrm{pcf}(e_k^{(+)}) < \aleph_{\sigma+\rho^{+4}}$, then $\max\ \mathrm{pcf}(e_k^{(+)}) < \aleph_{\sigma+\alpha_{i+1}}$.

So $D = \{\alpha_i \mid i < \rho^{+3}\}$ is defined. Let $\delta = \sup\ D$. Then $\mu = \aleph_{\sigma+\delta}$ is a singular cardinal of uncountable cofinality (that is, of cofinality ρ^{+3}). The Representation Theorem (Exercise 4.17) can be applied now. So there exists a closed unbounded set $C \subseteq D$ such that

$$\mu^+ = \max\ \mathrm{pcf}(\{\aleph^+_{\sigma+\alpha} \mid \alpha \in C\}). \tag{I.43}$$

The closed unbounded set D is isomorphic to ρ^{+3}, and C is transformed under this isomorphism to a closed unbounded set $E \subseteq \rho^{+3}$. That is

$$E = \{i \in \rho^{+3} \mid \alpha_i \in C\}.$$

By the club-guessing property, there exists $k \in S$ such that $C_k \subset E$. If C'_k denotes the non-accumulation points of C_k, we claim that $B = \{\aleph^+_{\sigma+\alpha_j} \mid j \in C'_k\}$ has the (impossible) property excluded by Theorem 6.4. Since the order-type of C_k is ρ^+, that of C'_k is also ρ^+. It suffices to prove for every $i \in C_k$ that

$$\max\ \mathrm{pcf}(\{\aleph^+_{\sigma+\alpha_j} \mid j \in C_k \cap (i+1)\}) < \aleph_{\sigma+\alpha_{i+1}}. \tag{I.44}$$

Consider the definition of α_{i+1}. The set $e_k = \{\aleph_{\sigma+\alpha_j} \mid j \in C_k \cap (i+1)\}$ was defined, and since $e_k^{(+)} \subseteq \{\aleph^+_{\sigma+\alpha} \mid \alpha \in C\}$, (I.43) implies that $\max\ \mathrm{pcf}(e_k^{(+)}) \leq \mu^+$. So the answer to the question for e_k was "yes", and as a result (I.44) holds. $\dashv$

This theorem leads to surprising applications. Consider for example $A = \{\aleph_n \mid n \in \omega\}$. Then $\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) = \max \mathrm{pcf}(A)$ by Theorem 5.11. But $\mathrm{pcf}(A)$ is an interval of regular cardinals of size $< \aleph_4$. Hence if we write $\max \mathrm{pcf}(A) = \aleph_\alpha$, then $\alpha < \omega_4$. Thus

$$\mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq) < \aleph_{\omega_4}.$$

This result holds even if $2^{\aleph_0}$ is larger than $\aleph_{\omega_4}$. It follows now immediately that if $2^{\aleph_0} < \aleph_\omega$ then $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$. Shelah emphasizes that the former result (concerning the cofinality of $[\aleph_\omega]^{\aleph_0}$) is more basic, and hence one should ask questions concerning cofinalities rather than cardinalities, if one wants to get (absolute) answers.

Generalizing this, we have:

7.2 Theorem. *If $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$ then*

$$\mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_{(|\delta|^{+4})}.$$

Proof. Write $|\delta| = \kappa$. Then $\kappa < \aleph_\delta$ and if A is the interval of regular cardinals in $(\kappa, \aleph_\delta)$ then $|A| \leq |\delta| = \kappa$ and A is a progressive set. Theorem 5.11 applies with $\mu = \aleph_\delta$ and it yields

$$\mathrm{cf}([\aleph_\delta]^\kappa, \subseteq) = \max\ \mathrm{pcf}(A).$$

But A is an interval of regular cardinals, and hence $|\mathrm{pcf}(A)| < |A|^{+4}$, by Theorem 7.1. This implies that max $\mathrm{pcf}(A) < \aleph_{\delta+(|A|^{+4})} \leq \aleph_{|\delta|^{+4}}$. Hence $\mathrm{cf}([\aleph_\delta]^\kappa, \subseteq) < \aleph_{(|\delta|^{+4})}$. $\dashv$

We are now able to deduce the following application to cardinal arithmetic.

7.3 Theorem. *Suppose that δ is a limit ordinal and $|\delta|^{\mathrm{cf}(\delta)} < \aleph_\delta$. Then*

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{+4})}.$$

Proof. Since $|\delta|^{\mathrm{cf}(\delta)} < \aleph_\delta$, $\delta < \aleph_\delta$. It follows from the cofinality theorem above that

$$\aleph_\delta^{\mathrm{cf}(\delta)} \leq |\delta|^{\mathrm{cf}(\delta)} \cdot \mathrm{cf}([\aleph_\delta]^{|\delta|}, \subseteq) < \aleph_\delta \cdot \aleph_{(|\delta|^{+4})}. \tag{I.45}$$

$\dashv$

8. Revised GCH

The generalized continuum hypothesis (G.C.H) saying that $2^\kappa = \kappa^+$ for every (infinite) cardinal κ is readily seen to be equivalent to the statement that for every two regular cardinals $\kappa < \lambda$ we have $\lambda^\kappa = \lambda$. In [16] Shelah considers a "revised power set" operation $\lambda^{[\kappa]}$ defined as follows:

$$\lambda^{[\kappa]} = \min\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^{\leq\kappa} \text{ and } \forall u \in [\lambda]^\kappa \exists \mathcal{P}_0 \subseteq \mathcal{P}\ (|\mathcal{P}_0| < \kappa \wedge u = \bigcup \mathcal{P}_0)\}.$$

An inductive proof can show that the G.C.H. is equivalent to the statement that for all regular cardinals $\kappa < \lambda$, $\lambda^{[\kappa]} = \lambda$. The "revised" G.C.H theorem says that for "many" pairs of regular cardinals we have $\lambda^{[\kappa]} = \lambda$.

8.1 Theorem (Shelah's Revised G.C.H). *If θ is a strong limit uncountable cardinal, then for every $\lambda \geq \theta$, for some $\kappa_0 < \theta$, for every $\kappa_0 \leq \kappa < \theta$*

$$\lambda^{[\kappa]} = \lambda.$$

The proof that we give here is adopted from a later article ([13]) of Shelah, and it relies on two notions that we have to investigate first, $\mathrm{pcf}_{\sigma-com}(A)$ and $T_D(f)$.

8.2 Definition. Let $\lambda > \theta \geq \sigma = \mathrm{cf}(\sigma)$ be cardinals.

1. We say that $\mathcal{P} \subseteq [\lambda]^{\leq\theta}$ is a $(<\sigma)$-base for $[\lambda]^{\leq\theta}$ if every $u \in [\lambda]^{\leq\theta}$ is the union of fewer than σ members of $\mathcal{P}$. That is, for some $\mathcal{P}_0 \subseteq \mathcal{P}$, $|\mathcal{P}_0| < \sigma$, and $u = \bigcup \mathcal{P}_0$.

2. We define $\lambda^{[\sigma,\theta]} = \min\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^{\leq\theta}$ is a $(<\sigma)-$ base for $[\lambda]^{\leq\theta}\}$. Another notation for $\lambda^{[\sigma,\theta]}$ is $\lambda^{[\sigma,\leq\theta]}$. We have $\lambda^{[\sigma]} = \lambda^{[\sigma,\sigma]}$. In a similar fashion define $\lambda^{[\sigma,<\theta]}$. It is the minimal cardinality of a set $\mathcal{P} \subseteq [\lambda]^{<\theta}$ so that every $u \in [\lambda]^{<\theta}$ is a union of fewer than σ members of $\mathcal{P}$.

3. We say that $\mathcal{P} \subseteq [\lambda]^{\theta}$ is $(<\sigma)$-cofinal in $[\lambda]^{\theta}$ if every $u \in [\lambda]^{\theta}$ is included in the union of fewer than σ members of $\mathcal{P}$. That is, for some $\mathcal{P}_0 \subset \mathcal{P}$, $|\mathcal{P}_0| < \sigma$, and $u \subseteq \bigcup \mathcal{P}_0$.

4. We define $\lambda^{\langle\sigma,\theta\rangle} = \min\{|\mathcal{P}| \mid \mathcal{P} \subseteq [\lambda]^{\theta}$ is $(<\sigma)$-cofinal in $[\lambda]^{\theta}\}$. Define $\lambda^{\langle\sigma\rangle} = \lambda^{\langle\sigma,\sigma\rangle}$.

For a regular infinite cardinal σ and a set A of regular cardinal define

$$\begin{aligned}\mathrm{pcf}_{\sigma-com}(A) = \{\mathrm{tcf}(\Pi A/F) \mid &F \text{ is a } \sigma-\text{complete filter} \\ &\text{over } A \text{ and } \mathrm{tcf}(\Pi A/F) \text{ exists}\}.\end{aligned} \tag{I.46}$$

(A filter is σ complete if it is closed under the intersections of less than σ members of the filter.)

Clearly, $A \subseteq \mathrm{pcf}_{\sigma-com}(A) \subseteq \mathrm{pcf}(A)$.

Define $J^{\sigma-com}_{<\lambda}[A] \subseteq \mathcal{P}(A)$ by the formula $X \in J^{\sigma-com}_{<\lambda}[A]$ iff $X \subseteq A$ and whenever F is a σ-complete filter over A with $X \in F$ and such that $\mathrm{tcf}(\Pi A/F)$ exists, then $\mathrm{tcf}(\Pi A/F) < \lambda$. Equivalently,

$$J^{\sigma-com}_{<\lambda}[A] = \{X \subseteq A \mid \mathrm{pcf}_{\sigma-com}(X) \subseteq \lambda\}.$$

Clearly, $J_{<\lambda}[A] \subseteq J^{\sigma-com}_{<\lambda}[A]$

8.3 Lemma. $J^{\sigma-com}_{<\lambda}[A]$ *is a σ-complete ideal.*

Proof. Suppose that $X_i \in J^{\sigma-com}_{<\lambda}[A]$ for every $i < \sigma^*$ where $\sigma^* < \sigma$. We prove that $X = \bigcup_{i<\sigma^*} X_i \in J^{\sigma-com}_{<\lambda}[A]$. So let F be a σ-complete filter over A containing X and such that $\mathrm{tcf}(\Pi A/F) = \tau$ exists. We must show that $\tau < \lambda$. Assume that F is proper (the cofinality of a reduced product by a non-proper filter is 1). For every $i < \sigma^*$ consider the filter $F + X_i$ (defined as the collection of all subsets of A that contain a set of the form $A \cap X_i$ for $A \in F$). If for some $i < \sigma^*$, $F_i = F + X_i$ is proper, then it is a σ-complete filter containing X_i and such that $\mathrm{tcf}(\Pi A/F_i) = \tau$ (extending the filter F will not change the cofinality of the existing reduced product). But as $X_i \in J^{\sigma-com}_{<\lambda}[A]$, we get $\tau < \lambda$.

If, for every $i < \sigma^*$, $F + X_i$ is non-proper, then $X \setminus X_i \in F$. Hence the intersection of these sets which is the empty set is in F, and thus F is non-proper. $\dashv$

8.4 Lemma. *Suppose that A is a progressive set of regular cardinals and $\lambda = \max \mathrm{pcf}(A)$. Then $X \in J^{\sigma-com}_{<\lambda}[A]$ iff X is a union of $< \sigma$ members of $J_{<\lambda}[A]$. That is, $J^{\sigma-com}_{<\lambda}[A]$ is the σ-completion of $J_{<\lambda}[A]$.*

Proof. Let J be the σ-completion of $J_{<\lambda}[A]$. It is the collection of all sets that are union of fewer than σ members of $J_{<\lambda}[A]$. By the previous lemma, $J^{\sigma-com}_{<\lambda}[A]$ is σ-complete, and hence it contains J. It remains to prove that $J^{\sigma-com}_{<\lambda}[A] \subseteq J$. So no assumptions on A were needed in this direction.

Assume for a contradiction that $X \in J^{\sigma-com}_{<\lambda}[A] \setminus J$. Then $J + (A \setminus X)$, the ideal generated by J and $A \setminus X$, is proper. It is easily seen to be a σ-complete ideal. Let F be the dual filter of that ideal. Then F is σ-complete and $X \in F$. Hence the cofinality of $\Pi A/F$ is smaller than λ.

Since $\lambda = \max \mathrm{pcf} A$, there are f_ζ for $\zeta < \lambda$ that are increasing and cofinal in $\Pi A/J_{<\lambda}[A]$ (Exercise 4.3, or Theorem 4.13). But this sequence is also increasing and cofinal in $\Pi A/F$, and this is an obvious contradiction. $\dashv$

We now strengthen the lemma by removing the assumption that $\lambda = \max\ \mathrm{pcf}(A)$.

8.5 Theorem. *Let A be a progressive set of regular cardinals, and σ a regular cardinal. Then $J^{\sigma-com}_{<\lambda}[A]$ is the σ-completion of $J_{<\lambda}[A]$.*

Proof. We prove by induction on μ that for every progressive set A of regular cardinals with $\mu = \max \mathrm{pcf}(A)$, for all cardinals λ and σ (regular), $J^{\sigma-com}_{<\lambda}[A]$ is the σ-completion of $J_{<\lambda}[A]$.

We know already that $J_{<\lambda}[A] \subseteq J^{\sigma-com}_{<\lambda}[A]$ and that $J^{\sigma-com}_{<\lambda}$ is σ-complete. It remains to prove that any $X \in J^{\sigma-com}_{<\lambda}[A]$ is a union of less than σ sets from $J_{<\lambda}[A]$. If $\mu < \lambda$ then $X \in J_{<\lambda}[A]$ and this case is uninteresting. In case $\lambda \leq \mu$, $X \in J^{\sigma-com}_{<\mu}[A]$. So by the previous lemma, X is a union of less than σ sets from $J_{<\mu}[A]$. But the inductive assumption can be applied to each one of these sets, and the lemma follows since σ is regular. $\dashv$

Another characterization of the ideal $J^{\sigma-com}_{<\lambda}[A]$ is provided by the following theorem dealing with the cofinality of product of cardinals under the $<$ relation: $f < g$ iff for every $a \in \mathrm{dom}(f)$ $f(a) < g(a)$.

We know (Theorem 4.4) that $X \in J_{<\lambda}[A]$ iff $\mathrm{cf}(\Pi X) < \lambda$. For a similar characterization of $J^{\sigma-com}_{<\lambda}$ we need the following definition. Let σ be a regular cardinal and X a set of regular cardinals. If $\mathcal{F} \subseteq \Pi X$, we say that $\mathcal{F}$ is $(< \sigma)$-cofinal iff for every $f \in \Pi X$ there is a set $\mathcal{F}_0 \subseteq \mathcal{F}$ with $|\mathcal{F}_0| < \sigma$ and such that $f < \sup \mathcal{F}_0$. In other words, the functions formed by taking the supremum of fewer than σ functions from $\mathcal{F}$ form a cofinal set in ΠX. The $(< \sigma)$-cofinality of ΠX is the smallest cardinality of a $(< \sigma)$-cofinal subset. It makes sense to assume that $\sigma \leq \min X$ when inquiring about the $(< \sigma)$-cofinality of X.

8.6 Theorem. *Suppose that A is a progressive set of regular cardinals, $\sigma \leq \min A$ is a regular cardinal, and $\sigma \leq \mathrm{cf}(\lambda)$. Define*

$$J = \{B \subseteq A \mid B = \emptyset \text{ or } \Pi B \text{ has } (<\sigma)\text{-cofinality } < \lambda\}$$

Then $J = J^{\sigma-com}_{<\lambda}[A]$.

Proof. We first prove that $J \subseteq J^{\sigma-com}_{<\lambda}$. Suppose $B \in J$ and let D be a σ-complete filter over A containing B and such that $\mathrm{tcf}(\Pi A/D)$ exists and is equal to $\lambda' \geq \lambda$. This will lead to a contradiction, thereby proving that $B \in J^{\sigma-com}_{<\lambda}$. Since $\mathrm{tcf}(\Pi A/D) = \lambda'$, λ' is a regular cardinal and there is an increasing sequence S in $\Pi A/D$ of length λ' that is cofinal in $\Pi A/D$. By definition of $B \in J$, there is a set $\mathcal{F} \subseteq \Pi B$ of cardinality $< \lambda$ that is $(<\sigma)$-cofinal. For every $f \in \mathcal{F}$ there is a function $s \in S$ such that $f <_D s$ (f is defined on B and s on A, but as $B \in D$, this makes sense). Since λ' is regular and bigger than $|\mathcal{F}|$, there is a single $s \in S$ such that $f <_D s$ for every $f \in \mathcal{F}$. Since $\mathcal{F}$ is $(<\sigma)$-cofinal, $s <_D \sup \mathcal{F}_0$ for some $\mathcal{F}_0 \subseteq \mathcal{F}$ of size $< \sigma$. But as D is σ-complete, and $f <_D s$ for every $f \in \mathcal{F}_0$, $\sup \mathcal{F}_0 \leq_D s$ as well. This is a contradiction, and thus $J \subseteq J^{\sigma-com}_{<\lambda}[A]$.

Clearly $J_{<\lambda}[A] \subseteq J$ (by Theorem 4.4). If we prove that J is σ-complete then $J^{\sigma-com}_{<\lambda}[A] \subseteq J$ follows from the previous theorem.

So let $\sigma^* < \sigma$ and $X_i \in J$ for $i < \sigma^*$ be given. We shall prove that $X = \bigcup_{i<\sigma^*} X_i \in J$. For every $i < \sigma^*$ we have a $(<\sigma)$-cofinal set $P_i \subseteq \Pi X_i$ of cardinality $< \lambda$. Then $P = \bigcup_{i<\sigma^*} P_i$ has cardinality $< \lambda$ because $\sigma \leq \mathrm{cf}(\lambda)$. The domain of each function in P_i is X_i, but we can extend it arbitrarily on X and then P can be considered as a subset of ΠX. Clearly P is $(<\sigma)$-cofinal. ⊣

We shall apply this theorem to the ideal $J^{\sigma-com}_{\leq\lambda}[A]$ rather than $J^{\sigma-com}_{<\lambda}[A]$. That is, replacing λ with λ^+ in the theorem, we get the following corollary in which $\sigma \leq \mathrm{cf}\lambda$ is no longer required.

8.7 Corollary. *Suppose that A is a progressive set of regular cardinals, $\sigma \leq \min A$ is a regular cardinal, and $\sigma \leq \lambda$. Define*

$$J = \{B \subseteq A \mid B = \emptyset \text{ or } \Pi B \text{ has } (<\sigma)\text{-cofinality } \leq \lambda\}$$

Then $J = J^{\sigma-com}_{\leq\lambda}[A]$.

8.8 Theorem. *Suppose that:*

1. *$\lambda > \theta > \sigma > \aleph_0$ are given, where θ and σ are regular cardinals, and $2^{<\theta} \leq \lambda$.*
2. *For every $A \subseteq Reg \cap \lambda \setminus \theta$, if $|A| < \theta$ then $A \in J^{\sigma-com}_{\leq\lambda}[A]$.*

Then $\lambda = \lambda^{[\sigma,<\theta]}$.

Proof. Fix χ sufficiently large, and let $M \prec H(\chi)$ be an elementary substructure of cardinality λ and such that $\lambda + 1 \subset M$. We shall prove the following claim which yields the theorem:

$$M \cap [\lambda]^{<\theta} \text{ is a } (<\sigma)\text{-base for } [\lambda]^{<\theta}.$$

For this, we need the following lemma.

8.9 Lemma. *With the same assumptions of the theorem and on M, let $g : \kappa \to \lambda$ and $f : \kappa \to \lambda + 1$ be given with $\kappa < \theta$, $f \in M$, and such that $\forall a \in \kappa\ g(a) \leq f(a)$. Then there is a collection $\Phi \subseteq M$ of functions from κ to λ such that the following hold:*

1. *$|\Phi| < \sigma$.*
2. *For every $p \in \Phi$, $g \leq p \leq f$ (that is, for all $a \in \kappa$, $g(a) \leq p(a) \leq f(a)$).*
3. *For every $a \in \kappa$, if $g(a) < f(a)$, then for some $p \in \Phi$ $g(a) \leq p(a) < f(a)$.*

Proof. Think of f as an "approximation from above" in M to the function g (which is not in M, or else the theorem is trivial). The set Φ is not required to be a member of M, and each function of Φ (if different from f) is a better approximation that lies in M. For each $a \in \kappa$, if $f(a)$ is not the best approximation, then Φ contains a function that gets a better value at a.

Fix in M a sequence $\langle C_\delta \mid \delta \leq \lambda,\ \delta \in \lim \lambda \rangle$ such that $C_\delta \subseteq \delta$ is unbounded in δ and of order-type $\mathrm{cf}(\delta)$.

Define the following subsets of κ:

$$E_0 = \{a < \kappa \mid g(a) = f(a)\}$$

$$E_1 = \{a < \kappa \mid g(a) < f(a),\ f(a) \text{ is a successor ordinal}\}$$

$$E_2 = \{a < \kappa \mid g(a) < f(a), f(a) \text{ is a limit and } \mathrm{cf}(f(a)) < \theta\}$$

$$E_3 = \kappa \setminus (E_0 \cup E_1 \cup E_2).$$

Since $2^{<\theta} \leq \lambda$, any bounded subset of θ is in M. So each E_ℓ is in M. We define h on κ as follows. For $a \in E_0$, $h(a) = f(a)$. For $a \in E_1$, $h(a) + 1 = f(a)$. For $a \in \kappa \setminus (E_0 \cup E_1)$, $h(a) = \min C_{f(a)} \setminus g(a)$.

Obviously $h \restriction E_0 \cup E_1 \in M$. We prove that $h \restriction E_2 \in M$ as well. By definition $h \restriction E_2$ is a function in $\Pi_{\delta \in E_2} C_{f(\delta)}$. But θ is regular, and since $|E_2| < \theta$ and $\mathrm{cf}(f(\delta)) < \theta$, there is a bound below θ on the values of $\{\mathrm{cf}(f(a)) \mid a \in E_2\}$, and hence $|\Pi_{\delta \in E_2} C_{f(\delta)}| \leq 2^{<\theta} \leq \lambda$.

So $\Pi_{\delta \in E_2} C_{f(\delta)} \subset M$, and hence $h \restriction E_2 \in M$.

There is no reason to assume that $h \restriction E_3$ is in M, but we shall find a set Φ of size $< \sigma$ as required by the lemma. Define $A = \{\operatorname{cf} f(a) \mid a \in E_3\}$. Then $A \subseteq \lambda+1 \setminus \theta$ is a set of regular cardinals of size $\leq \kappa$, and so $A \in J^{\sigma-com}_{\leq\lambda}[A]$. There is by Corollary 8.7 a family $\mathcal{F}$ of size $\leq \lambda$ that is $(<\sigma)$-cofinal in ΠA. Since $A \in M$ we can have $\mathcal{F} \in M$ and $\mathcal{F} \subset M$. Since $\kappa < \min A$ and $A \subset Reg$, $\mathcal{F}$ yields a family of functions, $\mathcal{F}' \subset \Pi_{\delta\in E_3} C_{f(\delta)} = P$ that is $(<\sigma)$-cofinal in P. As $h \restriction E_3 \in P$, there is a set $\mathcal{F}_0 \subseteq \mathcal{F}'$ of size $< \sigma$ such that $h \restriction E_3 < \sup \mathcal{F}_0$. If $e \in \mathcal{F}_0$, then $e(\delta) < f(\delta)$ but $e(\delta) < g(\delta)$ is possible. So we correct each $e \in \mathcal{F}_0$ and define:

$$e'(\delta) = \begin{cases} e(\delta) & \text{if } g(\delta) \leq e(\delta), \\ f(\delta) & \text{otherwise.} \end{cases}$$

Then $e' \in M$ because $e, f \in M$ and every subset of κ is in M. The collection $\{h \restriction (E_0 \cup E_1 \cup E_2)^\frown e' \mid e \in \mathcal{F}_0\}$ is as required, and the lemma is proved. $\dashv$

We continue now with the proof of the theorem. So let $u \in [\lambda]^{<\theta}$ be given and we shall find a subset of $M \cap [\lambda]^{<\theta}$ of cardinality $< \sigma$ whose union is u. Let $\kappa = |u| < \theta$ be the cardinality of u and take an enumeration $g : \kappa \to u$. We shall define by induction on $n \in \omega$ a set Φ_n of functions from κ to λ such that the following holds.

1. Let $f_0 : \kappa \to \lambda + 1$ be defined by $f_0(a) = \lambda$. Then $\Phi_0 = \{f_0\}$.
2. For every n, $\Phi_n \subset M$ and $|\Phi_n| < \sigma$. If $f \in \Phi_n$ then $g \leq f$.
3. For every $f \in \Phi_n$ and $a \in \kappa$ such that $g(a) < f(a)$ there exists $p \in \Phi_{n+1}$ such that $g(a) \leq p(a) < f(a)$.

This is easily obtained by the lemma.

Let $\Phi = \bigcup_{n<\omega} \Phi_n$. Then $|\Phi| < \sigma$. For any $f \in \Phi$, the set

$$E(f) = \{f(a) \mid a \in \kappa \text{ and } f(a) = g(a)\}$$

is in M (because f is, and any subset of κ). We have $u = \cup\{E(f) \mid f \in \Phi\}$ because if $x \in u$ then $x = g(a)$ for some $a \in \kappa$, and $g(a) < f_0(a)$. There exists a sequence $f_n \in \Phi_n$ so that if $g(a) < f_n(a)$ then $f_{n+1}(a) < f_n(a)$. And necessarily for some n $g(a) = f_n(a)$. So $a \in E(f_n)$. This ends the proof of Theorem 8.8. $\dashv$

The following corollary shows that the theorem above can also be applied when $\operatorname{cf}(\theta) < \sigma$.

8.10 Corollary. *Suppose that:*

1. $\lambda > \theta > \sigma = \operatorname{cf}(\sigma) > \aleph_0$ *are given, where* $\operatorname{cf}(\theta) < \sigma$, *and* $2^{<\theta} \leq \lambda$.
2. *For every* $A \subseteq Reg \cap \lambda + 1 \setminus \theta$, *if* $|A| < \theta$ *then* $A \in J^{\sigma-com}_{\leq\lambda}[A]$.

Then $\lambda = \lambda^{[\sigma, \leq\theta]}$.

Proof. Fix a sequence $\langle \theta_i \mid i < \mathrm{cf}(\theta)\rangle$ of regular cardinals that is cofinal in θ and such that $\sigma < \theta_i$ for all i. We claim for every $i < \mathrm{cf}(\theta)$ that the assumptions of Theorem 8.8 hold for $\lambda > \theta_i > \sigma$, and hence $\lambda = \lambda^{[\sigma, <\theta_i]}$ follows. But this clearly implies that $\lambda = \lambda^{[\sigma, \leq\theta]}$.

For the claim, we must prove that if $\theta' < \theta$ is regular then for every $A \subseteq Reg \cap \lambda + 1 \setminus \theta'$, if $|A| < \theta'$ then $A \in J^{\sigma-com}_{\leq\lambda}[A]$. Suppose for a contradiction that this is not the case, and for some σ-complete filter D over $A \subset Reg \cap \lambda + 1 \setminus \theta'$ we have $\mathrm{tcf}(\Pi A/D) = \lambda_0 > \lambda$. We may assume that $A \subset \theta$, that is, we may assume that $A \cap \theta \in D$, or else $A \setminus \theta$ is not D-null and then it can be added to D without changing the true cofinality of the reduced product, which contradicts the assumptions of the theorem.

If for every $i < \mathrm{cf}(\theta)$ $A \setminus \theta_i \in D$, then by the σ-completeness of D and the fact that $\mathrm{cf}(\theta) < \sigma$, we get a contradiction. So for some i $A \cap \theta_i$ is not D-null. But then $D' = D + A \cap \theta_i$ is σ-complete and it follows that the true cofinality of $\Pi A/D'$ remains λ_0. Yet this is impossible since $(\theta_i)^{|A\cap\theta_i|} \leq 2^{<\theta} \leq \lambda$. $\dashv$

8.1. $T_D(f)$

Let J be an ideal over a cardinal κ. We recall some definitions. The collection of positive sets is denoted J^+. The corresponding dual filter is denoted J^*. If R is a relation, if f and g are functions defined on κ, then we define $f\ R_J\ g$ if and only if $\{i \in \kappa \mid f(i)\ R\ g(i)\} \in J^*$. We also write $f\ R_{J^+}\ g$ for $\{i \in \kappa \mid f(i)\ R\ g(i)\} \notin J$. That is, $f(i)\ R\ g(i)$ occurs positively.

Thus $f \neq_J g$ means that $\{i \in \kappa \mid f(i) = g(i)\} \in J$, and $f =_{J^+} g$ means that $\neg f \neq_J g$.

Let κ be a cardinal and D a filter over κ. Consider the $<_D$ ordering on On^κ. For $f \in \mathrm{On}^\kappa$, $\Pi_{i<\kappa} f(i)$ is denoted Πf, and $\Pi_{i<\kappa} f(i)/D$ is denoted $\Pi f/D$. (We consider only functions f such that $f(i) > 0$ for $i \in \kappa$.)

For $\mathcal{F} \subset \Pi f$, we say that $\mathcal{F}$ is a set of pairwise "D-different" functions, if for every distinct $f_1, f_2 \in \mathcal{F}$ we have $f_1 \neq_D f_2$. For any $f \in \mathrm{On}^\kappa$, define

$$T_D(f) = \sup\{|\mathcal{F}| \mid \mathcal{F} \subseteq \Pi f \text{ is a set of pairwise } D\text{-different functions}\}$$

(Shelah investigate several different definitions, and this cardinal is denoted T^0_D in [13].)

8.11 Theorem. *Suppose that D is a filter over κ, $f \in \mathrm{On}^\kappa$ and $T_D(f) = \lambda$. If $2^\kappa < \lambda$ then the supremum in the definition of $T_D(f)$ is attained. In fact, if $2^\kappa < \lambda$ and $\mathcal{F} \subset \Pi f$ is any maximal family of pairwise D-different functions, then $|\mathcal{F}| = \lambda$.*

Proof. Suppose on the contrary that $\mathcal{F} \subset \Pi f$ is maximal but $|\mathcal{F}| < \lambda$. Let $\mathcal{G} \subset \Pi f$ be a collection of pairwise D-different functions such that

$$|\mathcal{G}| > |\mathcal{F}| + 2^\kappa. \tag{I.47}$$

For every $g \in \mathcal{G}$ we can find $f = f(g) \in \mathcal{F}$ such that $X(g) = \{i \in \kappa \mid f(i) = g(i)\}$ is not D-null. As (I.47), there are two distinct functions g_1 and g_2 in $\mathcal{G}$ such that $f(g_1) = f(g_2)$ and $X(g_1) = X(g_2)$. But this implies that g_1 and g_2 agree on a non-null set which is a contradiction to the assumption that the functions in $\mathcal{G}$ are pairwise D-different. $\dashv$

An obvious observation which turns out to be crucial is the following.

8.12 Lemma. *If* $\mathrm{tcf}(\Pi f/D)$ *exists, then* $T_D(f) \geq \mathrm{tcf}(\Pi f/D)$.

Proof. If $\mathrm{tcf}(\Pi f/D) = \lambda$, then there exists a $<_D$ increasing sequence of length λ, and hence a set of cardinality λ of pairwise D-different functions. $\dashv$

Assume now that σ is a regular uncountable cardinal, and D is a σ-complete filter over κ. Then $\Pi f/D$ is well-founded. This is used in the following.

8.13 Lemma. *Suppose that σ is a regular uncountable cardinal and D is a σ-complete filter over κ. Suppose $f \in \mathrm{On}^\kappa$ and $T_D(f) \geq \lambda$ where $2^\kappa < \lambda$. Then for some $g \leq_D f$ we have $T_D(g) = \lambda$.*

Proof. Let $g \leq_D f$ be minimal in the $\leq_D$ ordering such that $T_D(g) \geq \lambda$. Suppose for a contradiction that $T_D(g) > \lambda$. There is a set $\{f_\alpha \mid \alpha < \lambda^+\}$ of pairwise D-different functions in Πg. For $\alpha < \lambda^+$ define

$$u_\alpha = \{\beta < \lambda^+ \mid f_\beta <_D f_\alpha\}.$$

If, for some α, $|u_\alpha| \geq \lambda$, then u_α proves that $T_D(f_\alpha) \geq \lambda$ in contradiction to the minimality of g. Hence $|u_\alpha| < \lambda$ for every $\alpha < \lambda^+$.

But now we can apply the Free Mapping theorem of Hajnal and obtain $F \subseteq \lambda^+$, of cardinality λ^+ such that $\alpha \notin u_\beta$ (and $\beta \notin u_\alpha$) for every $\alpha \neq \beta$ in F. (The argument in short is the following. First, we can find $\lambda_0 < \lambda$ such that $|u_\alpha| = \lambda_0$ for unboundedly many $\alpha < \lambda^+$. Re-enumerating, we may assume $|u_\alpha| = \lambda_0$ for every α. On those $\alpha < \lambda^+$ with cofinality λ_0^+ we bound $u_\alpha \cap \alpha$ in α, and use Fodor's lemma.)

Hence there are $f_\alpha \in \mathrm{On}^\kappa$ for $\alpha < (2^\kappa)^+$ such that $f_\alpha \not\leq_D f_\beta$ whenever $\alpha \neq \beta$. But this is impossible in view of the Erdos-Rado partition theorem $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$. Indeed, for $\alpha < \beta < \kappa^+$ define $h(\alpha, \beta)$ as some $i < \kappa$ such that $f_\beta(i) < f_\alpha(i)$. Then h has no infinite homogeneous set, which contradicts the Erdos-Rado theorem. Thus $T_D(g) = \lambda$.

Observe that since $2^\kappa < \lambda$, $L = \{a \in \kappa \mid g(a) \in \lim\}$ is not null in D, and hence we may assume without loss of generality that it is in D. (Or else let $h <_D g$ be such that $g(a) = h(a) + 1$ for every $a \notin L$, and $h(a) = g(a)$ on L. Let f_α, for $\alpha < \lambda$, exemplify $T_D(g) = \lambda$. By minimality of g, there are λ functions f_α that are equal to h on a positive subset of $\kappa \setminus L$. Since $2^\kappa < \lambda$, two such functions are equal on a positive set, which is impossible.) $\dashv$

The following is one of the two main arguments used in the proof of the revised GCH theorem.

8.14 Theorem. *Assume that $\lambda > \theta \geq \sigma = \mathrm{cf}(\sigma) > \kappa$ are cardinals such that:*

1. *$\theta^\kappa = \theta$.*
2. *If $\tau < \sigma$ then $\tau^\kappa < \sigma$.*
3. *J is an ideal on κ.*
4. *There is a sequence $\bar{\lambda} = \langle \lambda_i \mid i < \kappa \rangle$, $\lambda_i < \lambda$, such that*
   (a) $T_J(\bar{\lambda}) = \lambda$,
   (b) $\lambda_i^{\langle \sigma,\theta \rangle} = \lambda_i$ for every $i < \kappa$.

Then $\lambda^{\langle \sigma,\theta \rangle} = \lambda$. (If we also assume $2^\theta \leq \lambda$, then evidently $\lambda^{[\sigma,\theta]} = \lambda$.)

Proof. In the proof, we actually weaken the requirement $T_J(\bar{\lambda}) = \lambda$ to the following conjunction.

1. There are $f_\alpha \in \Pi_{i<\kappa}\lambda_i$, for $\alpha < \lambda$, such that $\alpha \neq \beta \longrightarrow f_\alpha \neq_J f_\beta$,
2. There are $g_\alpha \in \Pi_{i<\kappa}\lambda_i$, for $\alpha < \lambda$, such that for every $f \in \Pi_{i<\kappa}\lambda_i$ there exists $\alpha < \kappa$ with $f =_{J^+} g_\alpha$.

Fix a sequence of pairwise D-different functions $f_\alpha \in \Pi_{i<\kappa}\lambda_i$, for $\alpha < \lambda$, as in 1 above.

For every $i < \kappa$ we assume $\lambda_i^{\langle \sigma,\theta \rangle} = \lambda_i$, so there exists a family $\mathcal{P}_i \subseteq [\lambda_i]^\theta$ of cardinality λ_i that is $(< \sigma)$-cofinal in $[\lambda_i]^\theta$.

Since $|\mathcal{P}_i| = \lambda_i$, $\Pi_{i\in\kappa}\mathcal{P}_i$ is isomorphic to $\Pi_{i<\kappa}\lambda_i$. So there is (by 2 above) a family $\{g_\alpha \mid \alpha < \lambda\} \subset \Pi_{i\in\kappa}\mathcal{P}_i$ such that for every $g \in \Pi_{i\in\kappa}\mathcal{P}_i$ there is $\alpha < \lambda$ with $g =_{J^+} g_\alpha$.

For every $g \in \Pi_{i\in\kappa}\mathcal{P}_i$ and $A \in J^+$, let $g\restriction A$ be the restriction of g to A, and $\Pi g \restriction A$ is $\Pi_{i\in A}g(i)$. We define

$$\mathcal{F}(g\restriction A) = \{\zeta \in \lambda \mid \forall i \in A\ f_\zeta(i) \in g(i)\}.$$

In other words, $\mathcal{F}(g \restriction A)$ is the set of $\zeta \in \lambda$ such that $f_\zeta \restriction A \in \Pi g \restriction A$. Observe that if $A \subset B \subseteq \kappa$, then $\mathcal{F}(g \restriction A) \supseteq \mathcal{F}(g \restriction B)$.

8.15 Claim. *For every $g \in \Pi_{i\in\kappa}\mathcal{P}_i$ and $A \in J^+$, $|\mathcal{F}(g \restriction A)| \leq \theta$.*

Since $g(i) \in \mathcal{P}_i \subseteq [\lambda_i]^\theta$, $|\Pi_{i\in A}g(i)| \leq \theta^\kappa = \theta$. So, if $|\mathcal{F}(g \restriction A)| > \theta$, we would have $\zeta \neq \zeta'$ in λ with $f_\zeta \restriction A = f_{\zeta'} \restriction A$. But as $A \in J^+$, this contradicts $f_\zeta \neq_J f_{\zeta'}$ and proves the claim.

8.16 Claim. *Every $u \in [\lambda]^\theta$ is included in a union of fewer than σ sets of the form $\mathcal{F}(g_\alpha \restriction A)$. That is, the collection $\mathcal{F} = \{\mathcal{F}(g_\alpha \restriction A) \mid \alpha < \lambda, A \in J^+\}$ is $(< \sigma)$-cofinal in $[\lambda]^\theta$.*

Observe first that as $|\mathcal{F}| \leq \lambda \cdot 2^\kappa = \lambda$, this claim proves the theorem.
Given $u \in [\lambda]^\theta$ define for every $i < \kappa$

$$u_i = \{f_\alpha(i) \mid \alpha \in u\}.$$

Then $u_i \in [\lambda_i]^{\leq\theta}$ and hence there is $\mathcal{P}_i^u \subset \mathcal{P}_i$ with $|\mathcal{P}_i^u| < \sigma$ and such that $u_i \subseteq \bigcup \mathcal{P}_i^u$. Since σ is regular, some $\tau < \sigma$ bounds all the cardinals $\sigma_i = |\mathcal{P}_i^u|$, and, as $\tau^\kappa < \sigma$, we have that $|\Pi_{i\in\kappa}\sigma_i| < \sigma$. So

$$\mathcal{G} = \Pi_{i\in\kappa}\mathcal{P}_i^u$$

is a subset of $\Pi_{i\in\kappa}\mathcal{P}_i$ of size $< \sigma$. The following two lemmas finish the proof of our claim.

8.17 Lemma. $u \subseteq \bigcup\{\mathcal{F}(g) \mid g \in \mathcal{G}\}$.

Proof. If $\zeta \in u$ then $f_\zeta(i) \in u_i$ for every $i \in \kappa$. Thus $f_\zeta(i) \in \bigcup \mathcal{P}_i^u$ for every $i < \kappa$, and we can find $g \in \mathcal{G}$ such that $f_\zeta(i) \in g(i)$ for all $i < \kappa$. Namely, $\zeta \in \mathcal{F}(g)$ as required. ⊣

8.18 Lemma. *For every $g \in \mathcal{G}$ there is $\alpha < \lambda$ and $A \in J^+$ such that $\mathcal{F}(g) \subseteq \mathcal{F}(g_\alpha \restriction A)$. Thus as $|\mathcal{G}| < \sigma$, u is contained in the union of fewer than σ sets of the form $\mathcal{F}(g_\alpha \restriction A)$.*

Proof. For every $g \in \mathcal{G}$ there is some $\alpha < \lambda$ such that $g =_{J^+} g_\alpha$. That is, for some $A \in J^+$, $g \restriction A = g_\alpha \restriction A$. We already observed that $\mathcal{F}(g) \subseteq \mathcal{F}(g \restriction A)$, and hence the lemma follows. So Theorem 8.14 is proved. ⊣

We shall use a variant of Theorem 8.14 in which the assumption $\theta^\kappa = \theta$ is replaced with the assumption that θ is a strong limit cardinal with $\mathrm{cf}(\theta) < \sigma$.

8.19 Corollary. *Assume that $\lambda > \theta > \sigma = \mathrm{cf}(\sigma) > \kappa$ are cardinals such that:*

1. *θ is a strong limit cardinal and $\mathrm{cf}(\theta) < \sigma$.*
2. *If $\tau < \sigma$ then $\tau^\kappa < \sigma$.*
3. *J is an ideal on κ.*
4. *There is a sequence $\bar{\lambda} = \langle \lambda_i \mid i < \kappa \rangle$, $\lambda_i < \lambda$, such that*
   - *(a) $T_J(\bar{\lambda}) = \lambda$,*
   - *(b) $\lambda_i^{\langle \sigma,\theta \rangle} = \lambda_i$ for every $i < \kappa$.*

Then $\lambda^{\langle\sigma,\theta\rangle} = \lambda$.

Proof. Fix a cofinal in θ sequence $\langle\theta_\epsilon \mid \epsilon < \mathrm{cf}(\theta)\rangle$ such that $\theta_\epsilon^\kappa = \theta_\epsilon$ and $\sigma < \theta_\epsilon$ for every ϵ. (Start with any cofinal sequence, and replace θ_ϵ with $(\theta_\epsilon)^\kappa$ if necessary.)

Consider any $\epsilon < \mathrm{cf}(\theta)$. Observe that for every $i < \kappa$ we have $\lambda_i = \lambda_i^{[\sigma,\theta_\epsilon]}$. This follows immediately from the assumptions that θ is a strong limit cardinal with $\mathrm{cf}(\theta) < \sigma$, and such that $\lambda_i = \lambda_i^{\langle\sigma,\theta\rangle}$. Hence Theorem 8.14 is applicable (with θ_ϵ in the role of θ) and $\lambda = \lambda^{\langle\sigma,\theta_\epsilon\rangle}$. Since this holds for every $\epsilon < \mathrm{cf}(\theta)$, we get $\lambda = \lambda^{\langle\sigma,\theta\rangle}$. ⊣

8.2. Proof of the revised GCH

We prove the following form of the revised G.C.H.

8.20 Theorem. *If θ is a strong limit singular cardinal, then for every $\lambda \geq \theta$, for some $\sigma < \theta$,*

$$\lambda = \lambda^{[\sigma,\theta]}.$$

Proof. Let $\sigma_0 = (\mathrm{cf}\ \theta)^+$.

The theorem is proved by induction on λ. For $\lambda = \theta$, $\lambda = \lambda^{[\sigma_0,\theta]}$, and the family of all bounded subsets of θ is an evidence for this equality. (Any subset of θ is a union of $\mathrm{cf}(\theta)$ bounded subsets.)

We note for clarification that the induction can easily proceed in case $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\theta)$, and so we may assume that $\mathrm{cf}(\lambda) = \mathrm{cf}(\theta)$. However, we shall not make any use of this in the following proof.

Case 1: For every $A \subseteq Reg \cap \lambda \setminus \theta$, if $|A| < \theta$ then $A \in J^{\sigma_0-com}_{\leq\lambda}[A]$.

In this case the inductive assumption is dispensable and Corollary 8.10 yields immediately that $\lambda = \lambda^{[\sigma_0,\leq\theta]}$.

Case 2: Not Case 1:

For some $A \subseteq Reg \cap \lambda \setminus \theta$ with $|A| < \theta$, $A \notin J^{\sigma_0-com}_{\leq\lambda}[A]$.

Hence there is a σ_0-complete filter D over A, where $|A| = \kappa < \theta$, such that $\mathrm{tcf}(\Pi A/D) > \lambda$. Say $f : \kappa \to A$ enumerates A. By Lemma 8.12, $T_D(f) \geq \mathrm{tcf}(\Pi A/D) > \lambda$. By Lemma 8.13, there exists $g \leq f$ defined over κ so that $T_D(g) = \lambda$.

We claim that $\{i < \kappa \mid g(i) \geq \theta\} \in D$. If not, then $\{i < \kappa \mid g(i) < \theta\}$ is D-positive. But since $\mathrm{cf}(\theta) < \sigma_0$ and D is σ_0-complete, there is $\theta' < \theta$ so that $X = \{i < \kappa \mid g(i) < \theta'\}$ is D-positive. Hence $T_D(g \restriction X) = \lambda$. But this is impossible since θ is strong limit and $(\theta')^\kappa < \theta$.

So we can assume now that for every $i < \kappa$, $g(i) \geq \theta$. Hence by the inductive assumption there is $\sigma(i) < \theta$ so that

$$g(i) = g(i)^{[\sigma(i),\theta]}. \tag{I.48}$$

Since $\mathrm{cf}(\theta) < \sigma_0$ and D is σ_0-complete, there is σ, such that $\kappa, \sigma_0 < \sigma < \theta$ and $\{i < \kappa \mid \sigma(i) < \sigma\}$ is D-positive. For notational simplicity we assume that $\sigma(i) < \sigma$ for all $i < \kappa$. Take $\sigma_1 = (\sigma^\kappa)^+$. Now apply Corollary 8.19 to $\lambda > \theta > \sigma_1 > \kappa$. This yields $\lambda^{\langle\sigma_1,\theta\rangle} = \lambda$, but since θ is a strong limit cardinal with $\mathrm{cf}(\theta) < \sigma_1$ we obtain $\lambda^{[\sigma_1,\theta]} = \lambda$.

⊣

We note that Theorem 8.1 did not make the assumption that θ is a singular cardinal, but Theorem 8.20 did. To see how 8.1 can be derived from 8.20, we argue as follows in case θ is a regular uncountable strong limit cardinal. There is a stationary set $S \subset \theta$ of strong limit singular cardinals. So if $\lambda \geq \theta$, then Theorem 8.20 applies to each $\theta' \in S$, and $\lambda = \lambda^{[\sigma(\theta'),\theta']}$ follows for some $\sigma(\theta') < \theta'$. By Fodor's theorem, there is fixed $\sigma < \theta$ such that $\sigma = \sigma(\theta')$ for a stationary set of cardinals $\theta' \in S$. This gives $\lambda = \lambda^{[\sigma,<\theta]}$. So obviously for every $\sigma \leq \kappa < \theta$, we get $\lambda = \lambda^{[\kappa]}$.

8.3. Applications of the revised GCH

Two applications are given here, the first to the existence of diamond sequences and the second to cellularity of Boolean algebras. Both use the following immediate corollary of the revised GCH theorem.

> If $\alpha \geq \beth_\omega$ then for some regular uncountable $\sigma < \beth_\omega$ there is a collection $P_\alpha \subseteq [\alpha]^\sigma$ where $|P_\alpha| = |\alpha|$ and such that for each $x \in [\alpha]^\sigma$, for some $p \in P_\alpha$, $p \subseteq x$. (I.49)

To begin this section we recall that for a stationary set $S \subseteq \lambda^+$, $\diamondsuit^-_{\lambda^+}(S)$ is the following diamond statement: there is a sequence $\langle S_\alpha \mid \alpha \in S\rangle$ where $S_\alpha \subseteq \mathcal{P}(\alpha)$, $|S_\alpha| \leq \lambda$, and for every $A \subseteq \lambda^+$, $\{\alpha \in S \mid A \cap \alpha \in S_\alpha\}$ is a stationary set. If $|S_\alpha| = 1$, that is essentially $S_\alpha \subseteq \alpha$, then the sequence is the usual diamond sequence on S, and the resulting statement is the classical diamond $\diamondsuit_{\lambda^+}(S)$. An intriguing theorem of Kunen's (see [11]) states that $\diamondsuit^-_{\lambda^+}(S)$ is equivalent to $\diamondsuit_{\lambda^+}(S)$. (Somewhat more generally, this holds for an arbitrary regular cardinal μ not necessarily a successor cardinal, where $\diamondsuit^-_\mu(S)$ is the diamond statement obtained by restricting S_α to have cardinality not greater than that of α.) When $S = \lambda^+$, we write $\diamondsuit^-_{\lambda^+}$ instead of $\diamondsuit^-_{\lambda^+}(S)$ etc.

A beautiful argument of Gregory [4] proves that if $2^\lambda = \lambda^+$ and $\lambda^{\aleph_0} = \lambda$, then $\diamondsuit^-_{\lambda^+}(S^{\lambda^+}_\omega)$ where $S^{\lambda^+}_\omega$ is the stationary set of ordinals in λ^+ of cofinality ω. (There are stronger formulations, but this suffices to demonstrate the application we have in mind.) To prove this theorem, let $\{X_i \mid i < \lambda^+\}$

be an enumeration of all bounded subsets of λ^+. For every $\alpha < \lambda^+$ define S_α as the collection of all subsets of α that are formed by taking countable unions of sets from $\{X_i \mid i < \alpha\}$. Since $|\alpha|^{\aleph_0} \leq \lambda$, $|S_\alpha| \leq \lambda$. Now, if $A \subseteq \lambda^+$ is given, then the set, C, of $\alpha < \lambda^+$ for which $\forall \zeta < \alpha\ \exists i < \alpha\ (A \cap \zeta = X_i)$ is closed unbounded in λ^+. If $\alpha \in C$ and $\mathrm{cf}(\alpha) = \omega$ then $A \cap \alpha \in S_\alpha$. Applying Kunen's theorem, we can obtain $\diamondsuit_{\lambda^+}(S^{\lambda^+}_\omega)$.

The revised GCH enables in many cases a stronger theorem in which $\lambda^{\aleph_0} = \lambda$ is not required.

8.21 Theorem. *If $\lambda \geq \beth_\omega$ and $2^\lambda = \lambda^+$, then $\diamondsuit_{\lambda^+}$ holds. (Hence $\diamondsuit_{\lambda^+}$ is in fact equivalent to $2^\lambda = \lambda^+$ for every $\lambda \geq \beth_\omega$.)*

Proof. As before, let $\{X_i \mid i < \lambda^+\}$ enumerate all bounded subsets of λ^+. $\beth_\omega$ is the first strong limit cardinal, and the revised GCH theorem applies to $\lambda \geq \beth_\omega$. So there is $\sigma < \beth_\omega$ such that (I.49) holds for some family $P \subseteq [\lambda]^\sigma$.

For every α in the interval $[\lambda, \lambda^+)$, $|\alpha| = \lambda$ and hence P can be transformed into a family $P_\alpha \subset [\alpha]^\sigma$ such that (I.49) holds (same σ for all α's). Now we define S_α as the collection of all subsets of α obtained as unions of the form $\bigcup\{X_i \mid i \in B\}$ where $B \in P_\alpha$. So $|S_\alpha| \leq \lambda$.

The argument to prove that $\langle S_i \mid i < \lambda^+\rangle$ is a diamond sequence is now familiar. Let $A \subseteq \lambda^+$ be any set. There is a closed unbounded $C \subset \lambda^+$ as before so that for $\alpha \in C$ and $\zeta < \alpha$ there is $i < \alpha$ such that $A \cap \zeta = X_i$. Now pick any $\alpha \in C$ such that $\mathrm{cf}(\alpha) = \sigma$. Pick an increasing sequence $\langle \alpha_\epsilon \mid \epsilon < \sigma\rangle$ cofinal in α, and for each $\epsilon < \kappa$ find $i(\epsilon) < \alpha$ such that $A \cap \alpha_\epsilon = X_{i(\epsilon)}$. Define $u = \{i(\epsilon) \mid \epsilon < \sigma\}$. Observe that if $K \subseteq \sigma$ is any unbounded subset of σ then $\bigcup\{X_{i(\epsilon)} \mid \epsilon \in K\} = A \cap \alpha$. For some $B \in P_\alpha$, $i(\epsilon) \in B$ for unboundedly many $\epsilon < \sigma$. Hence $A \cap \alpha = \bigcup\{X_i \mid i \in B\} \in S_\alpha$. ⊣

We now begin the second application.

8.22 Definition. A subset X of a Boolean algebra is μ-linked if there is a function $h : X \to \mu$ such that $x \wedge y \neq 0_B$ whenever $h(x) = h(y)$.

Our aim is to prove the following theorem from [15]. (For background and motivation and additional results consult [15] and [5].)

8.23 Theorem. *Assume that $\mu = \mu^{<\beth_\omega}$. If B is a c.c.c. Boolean algebra of cardinality $\leq 2^\mu$, then B is μ-linked.*

The proof which follows is an example of an induction that relies on the revised GCH. Since B satisfies the countable chain condition, its completion has cardinality $\leq |B|^{\aleph_0} \leq 2^\mu$, and so we can assume that B is a complete Boolean algebra (and when we prove that it is μ-linked then the original algebra which is embedded in its completion is also μ-linked).

We prove by induction on λ, a cardinal such that $\mu \leq \lambda \leq 2^\mu$, that any subset of B of cardinality λ is μ-linked. This is obvious for $\lambda = \mu$, or when

$\mathrm{cf}(\lambda) \leq \mu$ (and the inductive claim holds for smaller cardinals), and so we may assume that $\mathrm{cf}(\lambda) > \mu$. There are several ingredients in the proof of this theorem, and so it is postponed until the required preparations are made.

8.24 Definition. Let C be a Boolean algebra, and $D \subseteq C$ a subalgebra. For any $x \in C$ let $F_x = \{d \in D \mid x \leq d\}$ be the filter generated by x. For a cardinal θ the following property is denoted $(**)_\theta$ (for the pair D and C):

$(**)_\theta$ For every $x \in C$ there is $F \subseteq F_x$ of cardinality $\leq \theta$ and such that for every $b \in F_x$ there is $a \in F$ such that $a \leq b$.

In other words, F_x is generated by a subset of cardinality $\leq \theta$.

8.25 Lemma. *Let θ, μ, and κ be cardinals such that $\theta, \mu \leq \kappa$. Suppose that C is a Boolean algebra with a decomposition $C = \bigcup_{\alpha<\kappa} C_\alpha$, where the sequence of Boolean subalgebras C_α is increasing and continuous (for limit δ, $C_\delta = \bigcup_{i<\delta} C_i$). Assume the following:*

1. *$C_0 = \emptyset$.*
2. *Each C_α is μ-linked.*
3. *Property $(**)_\theta$ holds for each of the pairs C_α, C.*

Let χ be a sufficiently large cardinal and consider the structure H_χ (with some well ordering of its universe, and with C and its decomposition as constants). Suppose that M_1 and M_2 are two elementary substructures of H_χ that are isomorphic with an isomorphism $g : M_1 \to M_2$ that is the identity on $\kappa \cap M_1 \cap M_2$. Suppose in addition that $\theta \subset M_1 \cap M_2$, and that $M_1 \cap \mu = M_2 \cap \mu$.

Then for every non-zero $x \in M_1 \cap C$,

$$x \wedge g(x) \neq 0_C.$$

Proof. The rank of an element $c \in C$ is the least ordinal τ such that $c \in C_\tau$. Since $C_0 = \emptyset$, the rank of c is a successor ordinal (below κ) such that $c \in C_{\alpha+1} \setminus C_\alpha$. Take $x \in M_1$ of minimal rank $\alpha+1$ such that $x \wedge g(x) = 0_C$ and we shall obtain a contradiction.

Case 1. $\alpha \in M_1 \cap M_2$. So $g(\alpha) = \alpha$. Let $h : C_{\alpha+1} \to \mu$ be the least function (in the well-ordering of H_χ) given by the assumption that $C_{\alpha+1}$ is μ-linked. So $h \in M_1 \cap M_2$, and since h is definable from α we have $g(h) = h$ (as $g(\alpha) = \alpha$). Say $h(x) = \eta \in \mu$. As $M_1 \cap \mu = M_2 \cap \mu$, we have $g(h(x)) = g(\eta) = \eta$. But $g(h(x)) = g(h)(g(x)) = h(g(x))$. So $h(g(x)) = \eta$, and hence $h(x) = h(g(x))$ which implies that x and $g(x)$ have non-zero meet in C.

Case 2. $\alpha \in M_1 \setminus M_2$, and hence $\alpha \neq g(\alpha)$ and $g(\alpha) \in M_2 \setminus M_1$. Suppose that $g(\alpha) < \alpha$ (case $g(\alpha) > \alpha$ is symmetric). Say $g(x) = y$, and $g(\alpha) = \beta$.

Then $\beta+1$ is the rank of y. Let $\alpha_1 \leq \alpha$ be the least ordinal in M_1 that is strictly above β. Since $\beta+1 \leq \alpha_1$,

$$y \in C_{\alpha_1}.$$

Let $F_x \subset C_{\alpha_1}$ be the filter generated by x. Property $(**)_\theta$ of the pair C_{α_1} and C implies the existence of $F \subseteq F_x$ of cardinality $\leq \theta$ that generates F_x. As x and y are disjoint, the complement, $-y$, of y is in F_x (since it is in C_{α_1}) and hence there is $a \in F$ that is disjoint from y. Since α_1 and x are in M_1, we have F_x and F in M_1 as well. But as θ is included in M_1, $F \subset M_1$ and hence $a \in M_1$ follows. The rank of a is $\alpha_2+1 \leq \alpha_1$. The minimality of α_1 implies that $\alpha_2 < \beta$ (equality is impossible because β is not in M_1). But now we can apply a similar argument to F_y (for the pair C_β, C) and discover $b \in C_\beta \cap M_2$ that is disjoint to a. Say $u \in M_1$ is such that $g(u)=b$. Then $u \in C_\alpha$ and hence $x_0 = u \wedge a$ is in C_α. Since $b \in F_y$, $u \in F_x$, and hence x_0 is in F_x too. In particular, $x_0 \neq 0_C$. But $g(x_0) = b \wedge g(a)$ and x_0 is disjoint to $b \wedge g(a)$ because already a is disjoint to b. So x_0 is disjoint to $g(x_0)$, in contradiction to the minimality of the rank of x. ⊣

Here is a lemma which is an immediate consequence of the Engelking and Karlowicz theorem [3]; we state it for reference and will return to its proof later on.

8.26 Lemma. *If $\mu^\theta = \mu$ then there is a map $\tau : [2^\mu]^\theta \to \mu$ such that if $\tau(M_1) = \tau(M_2)$ then M_1 and M_2 have the same order-type (as subsets of the ordinal 2^μ) and the order isomorphism $g : M_1 \to M_2$ is the identity on $M_1 \cap M_2$.*

8.27 Corollary. *Suppose that $\theta < \mu < \kappa \leq 2^\mu$ are cardinals such that $\mu^\theta = \mu$. Let C be a Boolean algebra of cardinality $\leq 2^\mu$, and suppose that $C = \bigcup_{\alpha<\kappa} C_\alpha$ where the C_α form an increasing and continuous sequence of subalgebras such that: $C_0 = \emptyset$, each C_α is μ-linked, and $(**)_\theta$ holds for each pair C_α, C. Then C is μ-linked.*

Proof. Let χ be sufficiently large and H_χ be the structure of sets of cardinality hereditarily less than χ, with a well-ordering of the universe and C as a constant. For every $a \in C$ find $M(a) \prec H_\chi$ of cardinality θ and such that $\theta \subset M(a)$. With each $M = M(a)$ we associate the following three parameters.

1. $M \cap \mu \in [\mu]^\theta$. So there are μ such parameters.
2. $\tau(M \cap 2^\mu)$, where $\tau : [2^\mu]^\theta \to \mu$ is the map from the lemma above.
3. The isomorphism type of $M(a)$ (with a as a parameter). Since $2^\theta \leq \mu$ there are $\leq \mu$ such types.

The map taking $a \in C$ to the three parameters associated with $M(a)$ proves that C is μ-linked. For if $M(a)$ and $M(b)$ have the same parameters then $a \wedge b \neq 0_C$ by the following argument. Let $g : M(a) \to M(b)$ be the isomorphism given by item 3. Then $g(a) = b$, and we plan to apply Lemma 8.25. This is possible because (1) $\tau(M(a) \cap 2^\mu) = \tau(M(b) \cap 2^\mu)$ implies that g is the identity on $2^\mu \cap M(a) \cap M(b)$, (2) $\theta \subset M(a) \cap M(b)$ by assumption, and (3) $M(a) \cap \mu = M(b) \cap \mu$ because this is the first of the three parameters. $\dashv$

We continue the inductive proof of Theorem 8.23. Recall that $\lambda \leq 2^\mu$, B is a complete c.c.c. Boolean algebra of cardinality $\leq 2^\mu$, and every subset of B of cardinality $< \lambda$ is μ-linked. Our aim is to prove that any $X \subseteq B$ of cardinality λ is μ-linked. We intend to use Corollary 8.27, and we must find $C \subseteq B$ with $X \subseteq C$ and such that the premises of 8.27 hold.

For every α such that $\beth_\omega \leq \alpha < \lambda$ we have a regular uncountable cardinal $\sigma(\alpha) < \beth_\omega$ and a family $P_\alpha \subseteq [\alpha]^{\sigma(\alpha)}$ such that (I.49) holds. Since $cf(\lambda) \neq \omega$ (in fact $\mathrm{cf}(\lambda) > \mu$) there is an unbounded set $E \subset \lambda$ such that for some fixed σ we have $\sigma = \sigma(\alpha)$ for every $\alpha \in E$. The symbols E and σ retain this meaning throughout the proof. We define $\theta = 2^{<\sigma}$.

8.28 Lemma. *Let $\chi > 2^\mu$ be sufficiently large. Suppose that δ is an ordinal and $\langle M_i \prec H_\chi \mid i < \delta \rangle$ is such that:*

1. $\mathrm{cf}(\delta) > \sigma$.
2. $B, E \in M_0$ *and* $\beth_\omega \subset M_0$.
3. $M_i \subset M_j$ *for* $i < j$ *and* $M_i \in M_{i+1}$.
4. $|M_i| < \lambda$, *and* $M_i \cap \lambda \in \lambda$.

Then for $M = \bigcup_{i<\delta} M_i$ and $B_0 = B \cap M$, $(**)_{2^{<\sigma}}$ holds for the pair B_0 and B.

Proof. Given $x \in B$ consider $F_x \subset B_0$, the filter of members of B_0 that are greater than x. We want to find $F \subseteq F_x$ of cardinality $\leq \theta = 2^{<\sigma}$ that generates F_x. We choose $a_\zeta \in F_x$ for $\zeta < \sigma$ by the following inductive procedure. Suppose that $A_\zeta = \{a_\epsilon \mid \epsilon < \zeta\}$ is already chosen. Let $G_\zeta = \{\wedge Z \mid Z \subseteq A_\zeta \text{ and } Z \in M\}$. So G_ζ is the collection of all elements of B that can be formed by taking meets of subsets of A_ζ that happen to be in M. Clearly $A_\zeta \subseteq G_\zeta \subseteq F_x$. Since $|A_\zeta| < \sigma$, $|G_\zeta| \leq 2^{<\sigma}$. If there exists $a \in F_x$ not covering any $b \in G_\zeta$, then let a_ζ be such a. If there is no such a, then the procedure stops and $F = G_\zeta$ is as required. We shall prove that the construction cannot proceed for every $\zeta < \sigma$. Suppose it does, and consider $A = \{a_\zeta \mid \zeta < \sigma\}$. Since $\mathrm{cf}(\delta) > \sigma$ there is $i < \delta$ with $A \subset M_i$. As $|M_i| < \lambda$ there is, already in M_{i+1} an ordinal $\alpha \in E$ such that $|M_i| < \alpha$. So $\alpha + 1 \subset M_{i+1}$ and hence also $P_\alpha \subset M_{i+1}$ (where $P_\alpha \subseteq [\alpha]^\sigma$ satisfies

(I.49)). Viewing the universe of M_i as a copy of an ordinal $< \alpha$, the set A is a subset of α of cardinality σ, and we have some $p \in P_\alpha$ such that $p \subseteq A$. Since $\beth_\omega \subset M_{i+1}$, each subset of p is also in M_{i+1}. It follows that for every $a_\zeta \in p$, $A_\zeta \cap p \in M$ and hence $a_\zeta \not\geq \wedge(A_\zeta \cap p)$. Thus $\wedge(A_\zeta \cap p) - a_\zeta \neq 0_B$ is a sequence of σ pairwise disjoint members of B, which contradicts the c.c.c. since σ is uncountable. ⊣

We can complete now the proof of Theorem 8.23. We are assuming that $\lambda \leq 2^\mu$, $\mathrm{cf}(\lambda) > \mu$, $\mu^{<\beth_\omega} = \mu$, and every subset of B of cardinality smaller than λ is μ-linked. A set $X \subseteq B$ of cardinality λ is given, which we want to show is μ-linked. Pick χ sufficiently large and define $M_i \prec H_\chi$, for $i < \mathrm{cf}(\lambda) = \kappa$, such that

1. M_i is increasing and continuous with i. $|M_i| < \lambda$, $\lambda \cap M_i \in \lambda$, and $M_i \in M_{i+1}$.
2. $B, X \in M_0$, $\mu + 1 \subset M_0$, $\beth_\omega \subset M_0$, and $X \subset M = \bigcup_{i<\kappa} M_i$.

We shall prove that $B \cap M$ is μ-linked, and hence that X is μ-linked. For any set R of ordinals, let $nacc(R)$ denotes those $\alpha \in R$ that are not accumulation points of R (for some $\beta < \alpha$ $R \cap (\beta, \alpha) = \emptyset$).

Let $R \subset \kappa$ be a closed unbounded set such that every $\alpha \in nacc(R)$ is a limit ordinal with $\mathrm{cf}(\alpha) > \sigma$. Then, for $\delta \in nacc(R)$, Lemma 8.28 applies to the sequence $\langle M_i \mid i < \delta \rangle$ and hence the pair $B \cap M_\delta$, B satisfies $(**)_\theta$ ($\theta = 2^{<\sigma}$). But, then it follows that $(**)_\theta$ holds for *every* $\delta \in R$ for the pair $B \cap M_\delta$, B. Because if $\mathrm{cf}(\delta) > \sigma$ then the lemma applies, and if $\mathrm{cf}(\delta) \leq \sigma$ then δ is a limit of $\leq \sigma$ non-accumulation points of R, and hence $(**)_\theta$ holds for $B \cap M_\delta$ by accumulating $\leq \sigma$ sets, each of cardinality $\leq \theta$.

Now let $\langle \rho_i \mid i < \kappa \rangle$ be an increasing and continuous enumeration of R, and define $C_i = B \cap M_{\rho_i}$, $C = B \cap M$. Then Corollary 8.27 applies with $\theta = 2^{<\sigma}$ and yields that $B \cap M$ is μ-linked. This proves Theorem 8.23.

For completeness we review the theorem of Engelking and Karlowicz that was used in the proof.

8.29 Theorem ([3]). *Assume that θ and μ are cardinals such that $\mu^\theta = \mu$. Then there are functions $f_\xi : 2^\mu \to \mu$, for $\xi < \mu$, such that if $A \subset 2^\mu$, $|A| \leq \theta$, and $f : A \to \mu$, then there is $\xi < \mu$ such that $f \subset f_\xi$.*

Proof. It is convenient for the proof to see 2^μ as the set of functions from μ to 2. A "template" is a triple (D, S, F) where $D \in [\mu]^\theta$, $S \subset 2^D$ and $|S| \leq \theta$ (S is a set of functions from D to 2), and $F : S \to \mu$. The number of possible templates is μ.

For any template $T = (D, S, F)$ we define f_T on 2^μ. If $\alpha \in 2^\mu$ and $\alpha \restriction D \in S$, then we define $f_T(\alpha) = F(\alpha \restriction D)$ (if $\alpha \restriction D \notin S$ then $f_T(\alpha)$ is any value).

Given any $A \subset 2^\mu$, $|A| \le \theta$, and $f : A \to \mu$, find $D \in [\mu]^\theta$ such that $\alpha_1 \restriction D \neq \alpha_2 \restriction D$ whenever $\alpha_1 \neq \alpha_2$ are in A. $S = \{a \restriction D \mid a \in A\}$. For every $s \in S$ there is a unique $a \in A$ such that $s = a \restriction D$ and we define $F(s) = f(a)$. Then $f \subset f_T$. $\dashv$

We can prove now Lemma 8.26. Clearly the map assigning to each $X \in [2^\mu]^\theta$ its order-type (in θ^+) ensures that two sets are isomorphic if they have the same value. The problem is to ensure that two isomorphic sets have an isomorphism that is the identity on their intersection. Given $X \in [2^\mu]^\theta$, let f_X be the collapsing map which assigns to each $x \in X$ the order-type of $x \cap X$. Then there is some $\xi < \mu$ such that $f_X \subset f_\xi$ (by the Engelking and Karlowicz theorem). Let's color X with ξ (say the first one). Now if X and Y in $[2^\mu]^\theta$ have the same order-type and the same color ξ, then the isomorphism of X onto Y is the identity on $X \cap Y$ since it is equal to $g_2^{-1} \circ g_1$ where $g_1 = f_\xi \restriction X$ and $g_2 = f_\xi \restriction Y$.

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