# Hausdorff towers and gaps

Piotr Borodulin-Nadzieja

Winterschool 2013, Hejnice

joint work with **David Chodounsky**

# Tower.

## Tower

We say that $(T_\alpha)_{\alpha<\omega_1}$ is a *tower* if

- $T_\alpha \subseteq \omega$ for each $\alpha$;
- $T_\alpha \setminus T_\beta$ is finite ($T_\alpha \subseteq^* T_\beta$) iff $\alpha < \beta$.

# Tower.

## Tower

We say that $(T_\alpha)_{\alpha<\omega_1}$ is a *tower* if

- $T_\alpha \subseteq \omega$ for each $\alpha$;
- $T_\alpha \setminus T_\beta$ is finite ($T_\alpha \subseteq^* T_\beta$) iff $\alpha < \beta$.

# Tower.

## Tower

We say that $(T_\alpha)_{\alpha<\omega_1}$ is a *tower* if

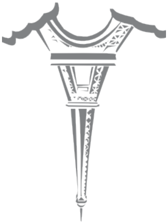
- $T_\alpha \subseteq \omega$ for each $\alpha$;
- $T_\alpha \setminus T_\beta$ is finite ($T_\alpha \subseteq^* T_\beta$) iff $\alpha < \beta$.

# Tower.

## Tower

We say that $(T_\alpha)_{\alpha<\omega_1}$ is a *tower* if

- $T_\alpha \subseteq \omega$ for each $\alpha$;
- $T_\alpha \setminus T_\beta$ is finite ($T_\alpha \subseteq^* T_\beta$) iff $\alpha < \beta$.



# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \not\subseteq T_\beta$ for each $\alpha < \beta$?

## Question 1 reformulated

Is there an uncountable family $\mathcal{T} \subseteq [\omega]^\omega$ such that

- $(\mathcal{T}, \subseteq^*)$ is well-ordered;
- $(\mathcal{T}, \subseteq)$ is an antichain?

# Question 1.

**Question 1**

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \not\subseteq T_\beta$ for each $\alpha < \beta$?

**Question 1 reformulated**

Is there an uncountable family $\mathcal{T} \subseteq [\omega]^\omega$ such that

- $(\mathcal{T}, \subseteq^*)$ is well-ordered;
- $(\mathcal{T}, \subseteq)$ is an antichain?

# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \nsubseteq T_\beta$ for each $\alpha < \beta$?

## Question 1 reformulated

Is there an uncountable family $\mathcal{T} \subseteq [\omega]^\omega$ such that

- $(\mathcal{T}, \subseteq^*)$ is well-ordered;
- $(\mathcal{T}, \subseteq)$ is an antichain?

# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \not\subseteq T_\beta$ for each $\alpha < \beta$?

## Question 1 reformulated

Is there an uncountable family $\mathcal{T} \subseteq [\omega]^\omega$ such that

- $(\mathcal{T}, \subseteq^*)$ is well-ordered;
- $(\mathcal{T}, \subseteq)$ is an antichain?

# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \nsubseteq T_\beta$ for each $\alpha < \beta$?

## Definition: **gap**

A family $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *gap* if

- $(L_\alpha)_\alpha$ and $(R_\alpha)_\alpha$ are towers;
- $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha$;
- there is no $L \subseteq \omega$ s.t. $L_\alpha \subseteq^* L$ and $R_\alpha \cap L =^* \emptyset$ for each $\alpha$.

# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \nsubseteq T_\beta$ for each $\alpha < \beta$?

## Definition: **gap**

A family $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *gap* if

- $(L_\alpha)_\alpha$ and $(R_\alpha)_\alpha$ are towers;
- $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha$;
- there is no $L \subseteq \omega$ s.t. $L_\alpha \subseteq^* L$ and $R_\alpha \cap L =^* \emptyset$ for each $\alpha$.

# Question 1.

**Question 1**

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \not\subseteq T_\beta$ for each $\alpha < \beta$?

**Definition: gap**

A family $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *gap* if

- $(L_\alpha)_\alpha$ and $(R_\alpha)_\alpha$ are towers;
- $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha$;
- there is no $L \subseteq \omega$ s.t. $L_\alpha \subseteq^* L$ and $R_\alpha \cap L =^* \emptyset$ for each $\alpha$.

# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \not\subseteq T_\beta$ for each $\alpha < \beta$?

## Definition: **gap**

A family $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *gap* if

- $(L_\alpha)_\alpha$ and $(R_\alpha)_\alpha$ are towers;
- $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha$;
- there is no $L \subseteq \omega$ s.t. $L_\alpha \subseteq^* L$ and $R_\alpha \cap L =^* \emptyset$ for each $\alpha$.

# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \not\subseteq T_\beta$ for each $\alpha < \beta$?

## Definition: **gap**

A family $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *gap* if

- $(L_\alpha)_\alpha$ and $(R_\alpha)_\alpha$ are towers;
- $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha$;
- there is no $L \subseteq \omega$ s.t. $L_\alpha \subseteq^* L$ and $R_\alpha \cap L =^* \emptyset$ for each $\alpha$.

# Question 1.

## Question 1

Is there a tower $(T_\alpha)_\alpha$ such that $T_\alpha \not\subseteq^* T_\beta$ for each $\alpha < \beta$?

## Definition: **Hausdorff gap**

A family $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *gap* if

- $(L_\alpha)_\alpha$ and $(R_\alpha)_\alpha$ are towers;
- $L_\alpha \cap R_\alpha = \emptyset$ for each $\alpha$;
- $\{\beta < \alpha \colon L_\beta \cap R_\alpha \subseteq n\}$ is finite for each $n \in \omega$ and $\alpha < \omega_1$.

# Answer to Question 1 (S. Todorčević).

Let $(L_\alpha, R_\alpha)_\alpha$ be a Hausdorff gap. Define $f$ ($dom(f) = \omega_1$):

$$f(\alpha) = \{\beta < \alpha \colon L_\beta \cap R_\alpha = \emptyset\}.$$

- Hausdorff condition $\implies$ $f\colon \omega_1 \to [\omega_1]^{<\omega}$;
- $f$ is a set-mapping (i.e. $\alpha \notin f(\alpha)$);
- Free-set theorem $\implies$ there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

$$\alpha \notin f(\beta) \text{ for each } \alpha, \beta \in \Lambda;$$

- If $L_\beta \cap R_\alpha \neq \emptyset$ then $L_\beta \setminus L_\alpha \neq \emptyset$;
- Finally: $L_\alpha \not\subseteq L_\beta$ for each $\alpha < \beta \in \Lambda$.

# Answer to Question 1 (S. Todorčević).

Let $(L_\alpha, R_\alpha)_\alpha$ be a Hausdorff gap. Define $f$ ($dom(f) = \omega_1$):

$$f(\alpha) = \{\beta < \alpha \colon L_\beta \cap R_\alpha = \emptyset\}.$$

- Hausdorff condition $\implies$ $f \colon \omega_1 \to [\omega_1]^{<\omega}$;
- $f$ is a set-mapping (i.e. $\alpha \notin f(\alpha)$);
- Free-set theorem $\implies$ there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

$$\alpha \notin f(\beta) \text{ for each } \alpha, \beta \in \Lambda;$$

- If $L_\beta \cap R_\alpha \neq \emptyset$ then $L_\beta \setminus L_\alpha \neq \emptyset$;
- Finally: $L_\alpha \not\subseteq L_\beta$ for each $\alpha < \beta \in \Lambda$.

# Answer to Question 1 (S. Todorčević).

Let $(L_\alpha, R_\alpha)_\alpha$ be a Hausdorff gap. Define $f$ ($dom(f) = \omega_1$):

$$f(\alpha) = \{\beta < \alpha \colon L_\beta \cap R_\alpha = \emptyset\}.$$

- Hausdorff condition $\implies$ $f \colon \omega_1 \to [\omega_1]^{<\omega}$;
- $f$ is a set-mapping (i.e. $\alpha \notin f(\alpha)$);
- Free-set theorem $\implies$ there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

$$\alpha \notin f(\beta) \text{ for each } \alpha, \beta \in \Lambda;$$

- If $L_\beta \cap R_\alpha \neq \emptyset$ then $L_\beta \setminus L_\alpha \neq \emptyset$;
- Finally: $L_\alpha \not\subseteq L_\beta$ for each $\alpha < \beta \in \Lambda$.

# Answer to Question 1 (S. Todorčević).

Let $(L_\alpha, R_\alpha)_\alpha$ be a Hausdorff gap. Define $f$ ($dom(f) = \omega_1$):

$$f(\alpha) = \{\beta < \alpha \colon L_\beta \cap R_\alpha = \emptyset\}.$$

- Hausdorff condition $\implies$ $f \colon \omega_1 \to [\omega_1]^{<\omega}$;
- $f$ is a set-mapping (i.e. $\alpha \notin f(\alpha)$);
- Free-set theorem $\implies$ there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

$$\alpha \notin f(\beta) \text{ for each } \alpha, \beta \in \Lambda;$$

- If $L_\beta \cap R_\alpha \neq \emptyset$ then $L_\beta \setminus L_\alpha \neq \emptyset$;
- Finally: $L_\alpha \not\subseteq L_\beta$ for each $\alpha < \beta \in \Lambda$.

# Answer to Question 1 (S. Todorčević).

Let $(L_\alpha, R_\alpha)_\alpha$ be a Hausdorff gap. Define $f$ ($dom(f) = \omega_1$):

$$f(\alpha) = \{\beta < \alpha \colon L_\beta \cap R_\alpha = \emptyset\}.$$

- Hausdorff condition $\implies$ $f \colon \omega_1 \to [\omega_1]^{<\omega}$;
- $f$ is a set-mapping (i.e. $\alpha \notin f(\alpha)$);
- Free-set theorem $\implies$ there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

$$\alpha \notin f(\beta) \text{ for each } \alpha, \beta \in \Lambda;$$

- If $L_\beta \cap R_\alpha \neq \emptyset$ then $L_\beta \setminus L_\alpha \neq \emptyset$;
- Finally: $L_\alpha \not\subseteq L_\beta$ for each $\alpha < \beta \in \Lambda$.

# Answer to Question 1 (S. Todorčević).

Let $(L_\alpha, R_\alpha)_\alpha$ be a Hausdorff gap. Define $f$ ($dom(f) = \omega_1$):

$$f(\alpha) = \{\beta < \alpha \colon L_\beta \cap R_\alpha = \emptyset\}.$$

- Hausdorff condition $\implies f \colon \omega_1 \to [\omega_1]^{<\omega}$;
- $f$ is a set-mapping (i.e. $\alpha \notin f(\alpha)$);
- Free-set theorem $\implies$ there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

$$\alpha \notin f(\beta) \text{ for each } \alpha, \beta \in \Lambda;$$

- If $L_\beta \cap R_\alpha \neq \emptyset$ then $L_\beta \setminus L_\alpha \neq \emptyset$;
- Finally: $L_\alpha \not\subseteq L_\beta$ for each $\alpha < \beta \in \Lambda$.

# Answer to Question 1 (S. Todorčević).

Let $(L_\alpha, R_\alpha)_\alpha$ be a Hausdorff gap. Define $f$ ($dom(f) = \omega_1$):

$$f(\alpha) = \{\beta < \alpha \colon L_\beta \cap R_\alpha = \emptyset\}.$$

- Hausdorff condition $\implies$ $f\colon \omega_1 \to [\omega_1]^{<\omega}$;
- $f$ is a set-mapping (i.e. $\alpha \notin f(\alpha)$);
- Free-set theorem $\implies$ there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

$$\alpha \notin f(\beta) \text{ for each } \alpha, \beta \in \Lambda;$$

- If $L_\beta \cap R_\alpha \neq \emptyset$ then $L_\beta \setminus L_\alpha \neq \emptyset$;
- Finally: $L_\alpha \not\subseteq L_\beta$ for each $\alpha < \beta \in \Lambda$.

# Answer.

**pre-Definition: Hausdorff tower**

A tower $(T_\alpha)_\alpha$ is Hausdorff if (it contains a subtower $(T'_\alpha)$ such that) for each $\alpha$ and $n$

$$\{\beta < \alpha \colon T_\beta \setminus T_\alpha \subseteq n\} \text{ is finite.}$$

**Proposition**

There is a Hausdorff tower. Each Hausdorff tower contains an uncountable $\subseteq$-antichain.

# Answer.

**Definition: Hausdorff tower**

A tower $(T_\alpha)_\alpha$ is Hausdorff if (it contains a subtower $(T'_\alpha)$ such that) for each $\alpha$ and $n$

$$\{\beta < \alpha \colon T'_\beta \setminus T'_\alpha \subseteq n\} \text{ is finite.}$$

**Proposition**

There is a Hausdorff tower. Each Hausdorff tower contains an uncountable $\subseteq$-antichain.

# Answer.

## Definition: **Hausdorff tower**

A tower $(T_\alpha)_\alpha$ is Hausdorff if (it contains a subtower $(T'_\alpha)$ such that) for each $\alpha$ and $n$

$$\{\beta < \alpha \colon T'_\beta \setminus T'_\alpha \subseteq n\} \text{ is finite.}$$

## Proposition

There is a Hausdorff tower. Each Hausdorff tower contains an uncountable $\subseteq$-antichain.

# Question 2.

**Question**

Is there a tower without an uncountable $\subseteq$-antichain?

**Theorem (Kunen, van Douwen, 1982)**

CH $\implies$ Yes.

# Question 2.

**Question**

Is there a tower without an uncountable $\subseteq$-antichain?

**Theorem (Kunen, van Douwen, 1982)**

CH $\implies$ Yes.

# Basic definitions.

**Definition: Hausdorff tower**

A tower $(T_\alpha)_\alpha$ is *Hausdorff* if (it contains a subtower $(T'_\alpha)$ such that) for each $\alpha$ and $n$

$$\{\beta < \alpha\colon T'_\beta \setminus T'_\alpha \subseteq n\} \text{ is finite.}$$

**Definition: Suslin tower**

A tower $(T_\alpha)_\alpha$ is *Suslin* if it does not contain an uncountable $\subseteq$-antichain.

**Definition: special tower**

A tower $(T_\alpha)_\alpha$ is *special* if it is not Suslin.

# Basic definitions.

**Definition: Hausdorff tower**

A tower $(T_\alpha)_\alpha$ is *Hausdorff* if (it contains a subtower $(T'_\alpha)$ such that) for each $\alpha$ and $n$

$$\{\beta < \alpha \colon T'_\beta \setminus T'_\alpha \subseteq n\} \text{ is finite.}$$

**Definition: Suslin tower**

A tower $(T_\alpha)_\alpha$ is *Suslin* if it does not contain an uncountable $\subseteq$-antichain.

**Definition: special tower**

A tower $(T_\alpha)_\alpha$ is *special* if it is not Suslin.

# Basic definitions.

**Definition: Hausdorff tower**

A tower $(T_\alpha)_\alpha$ is *Hausdorff* if (it contains a subtower $(T'_\alpha)$ such that) for each $\alpha$ and $n$

$$\{\beta < \alpha \colon T'_\beta \setminus T'_\alpha \subseteq n\} \text{ is finite.}$$

**Definition: Suslin tower**

A tower $(T_\alpha)_\alpha$ is *Suslin* if it does not contain an uncountable $\subseteq$-antichain.

**Definition: special tower**

A tower $(T_\alpha)_\alpha$ is *special* if it is not Suslin.

# Martin's Axiom (+ non-CH).

**Proposition**

$\mathrm{MA}(\omega_1) \implies$ each tower is Hausdorff.

**Proposition**

$\mathrm{MA}(\omega_1) \implies$ for each tower $(T_\alpha)_\alpha$ there is a tower $(T'_\alpha)_\alpha$ such that

- $|T_\alpha \setminus T'_\alpha| \leq 1$;
- $|T'_\alpha \setminus T_\alpha| \leq 1$;
- $(T'_\alpha)_\alpha$ is a $\subseteq$-antichain.

# Martin's Axiom (+ non-CH).

**Proposition**

MA($\omega_1$) $\implies$ each tower is Hausdorff.

**Proposition**

MA($\omega_1$) $\implies$ for each tower $(T_\alpha)_\alpha$ there is a tower $(T'_\alpha)_\alpha$ such that

- $|T_\alpha \setminus T'_\alpha| \leq 1$;
- $|T'_\alpha \setminus T_\alpha| \leq 1$;
- $(T'_\alpha)_\alpha$ is a $\subseteq$-antichain.

# Martin's Axiom (+ non-CH).

**Proposition**

MA($\omega_1$) $\implies$ each tower is Hausdorff.

**Proposition**

MA($\omega_1$) $\implies$ for each tower $(T_\alpha)_\alpha$ there is a tower $(T'_\alpha)_\alpha$ such that

- $|T_\alpha \setminus T'_\alpha| \leq 1$;
- $|T'_\alpha \setminus T_\alpha| \leq 1$;
- $(T'_\alpha)_\alpha$ is a $\subseteq$-antichain.

# Martin's Axiom (+ non-CH).

## Proposition

$MA(\omega_1) \implies$ each tower is Hausdorff.

## Proposition

$MA(\omega_1) \implies$ for each tower $(T_\alpha)_\alpha$ there is a tower $(T'_\alpha)_\alpha$ such that

- $|T_\alpha \setminus T'_\alpha| \leq 1$;
- $|T'_\alpha \setminus T_\alpha| \leq 1$;
- $(T'_\alpha)_\alpha$ is a $\subseteq$-antichain.

# Martin's Axiom (+ non-CH).

**Proposition**

$\mathrm{MA}(\omega_1) \implies$ each tower is Hausdorff.

**Proposition**

$\mathrm{MA}(\omega_1) \implies$ for each tower $(T_\alpha)_\alpha$ there is a tower $(T'_\alpha)_\alpha$ such that

- $|T_\alpha \setminus T'_\alpha| \leq 1$;
- $|T'_\alpha \setminus T_\alpha| \leq 1$;
- $(T'_\alpha)_\alpha$ is a $\subseteq$-antichain.

# OCA and PID.

**Proposition**

OCA $\implies$ each tower is special.

**Theorem**

Assume PID. Then each tower is Hausdorff if and only if $\mathfrak{b} > \omega_1$.

# OCA and PID.

**Proposition**

OCA $\implies$ each tower is special.

**Theorem**

Assume PID. Then each tower is Hausdorff if and only if $\mathfrak{b} > \omega_1$.

# OCA and PID.

## Proposition

OCA $\implies$ each tower is special.

## Theorem

Assume PID. Then each tower is Hausdorff if and only if $\mathfrak{b} > \omega_1$.

# Suslin towers (and oscillations).

**Proposition**

There exists a Suslin tower of size $\mathfrak{b}$. Consistently, there is a Suslin tower of size less than $\mathfrak{b}$.

**Theorem (Todorčević)**

If a tower generates a non-meager ideal, then it is Suslin.

# Suslin towers (and oscillations).

**Proposition**

There exists a Suslin tower of size $\mathfrak{b}$. Consistently, there is a Suslin tower of size less than $\mathfrak{b}$.

**Theorem (Todorčević)**

If a tower generates a non-meager ideal, then it is Suslin.

# Suslin towers (and oscillations).

**Proposition**

There exists a Suslin tower of size $\mathfrak{b}$. Consistently, there is a Suslin tower of size less than $\mathfrak{b}$.

**Theorem (Todorčević)**

If a tower generates a non-meager ideal, then it is Suslin.

# Suslin towers (and oscillations).

**Proposition**

There exists a Suslin tower of size $\mathfrak{b}$. Consistently, there is a Suslin tower of size less than $\mathfrak{b}$.

**Theorem (Todorčević)**

If a tower generates a non-meager ideal, then it is Suslin.

# Suslin tower from a Cohen real.

## Proposition

Let $\mathbb{C}$ be the Cohen forcing and let $(T_\alpha)_\alpha$ be a tower. Then

$$\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha \text{ is a Suslin tower.}$$

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Suslin tower from a Cohen real - proof.

- $\Vdash_{\mathbb{C}} (\dot{c} \cap T_\alpha)_\alpha$ is a tower;
- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_\alpha \cap n = T_\beta \cap n$;
- Let $m$ be such that $T_\alpha \setminus T_\beta \subseteq m$;
- Let $q|n = p$ and $q(i) = 0$ for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_\alpha \subseteq \dot{c} \cap T_\beta$.

# Special gaps.

## Special gap

A gap $(L_\alpha, R_\alpha)_\alpha$ is *special* if there is an uncountable $X \subseteq \omega_1$ such that

$$(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) \neq \emptyset$$

for each $\alpha < \beta \in X$.

## Oriented gap

A gap $(L_\alpha, R_\alpha)_\alpha$ is *oriented* if there is an uncountable $X \subseteq \omega_1$ such that

$$L_\alpha \cap R_\beta \neq \emptyset$$

for each $\alpha < \beta \in X$.

# Special gaps.

## Special gap

A gap $(L_\alpha, R_\alpha)_\alpha$ is *special* if there is an uncountable $X \subseteq \omega_1$ such that

$$(L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) \neq \emptyset$$

for each $\alpha < \beta \in X$.

## Oriented gap

A gap $(L_\alpha, R_\alpha)_\alpha$ is *oriented* if there is an uncountable $X \subseteq \omega_1$ such that

$$L_\alpha \cap R_\beta \neq \emptyset$$

for each $\alpha < \beta \in X$.

# Special gaps.

## Basic facts

- notation: "Hausdorff" = "equivalent to Hausdorff";
- Hausdorff $\Longrightarrow$ oriented $\Longrightarrow$ special;
- If $(L_\alpha, R_\alpha)_\alpha$ is Hausdorff, then $(L_\alpha)_\alpha$ is Hausdorff;
- If $(L_\alpha, R_\alpha)_\alpha$ is oriented, then $(L_\alpha)_\alpha$ is special.

## Questions - Scheepers (1993)

Is every oriented gap Hausdorff? Is every special gap oriented?

# Special gaps.

## Basic facts

- notation: "Hausdorff" = "equivalent to Hausdorff";
- Hausdorff $\implies$ oriented $\implies$ special;
- If $(L_\alpha, R_\alpha)_\alpha$ is Hausdorff, then $(L_\alpha)_\alpha$ is Hausdorff;
- If $(L_\alpha, R_\alpha)_\alpha$ is oriented, then $(L_\alpha)_\alpha$ is special.

## Questions - Scheepers (1993)

Is every oriented gap Hausdorff? Is every special gap oriented?

# Special gaps.

## Basic facts

- notation: “Hausdorff” = “equivalent to Hausdorff”;
- Hausdorff $\Longrightarrow$ oriented $\Longrightarrow$ special;
- If $(L_\alpha, R_\alpha)_\alpha$ is Hausdorff, then $(L_\alpha)_\alpha$ is Hausdorff;
- If $(L_\alpha, R_\alpha)_\alpha$ is oriented, then $(L_\alpha)_\alpha$ is special.

## Questions - Scheepers (1993)

Is every oriented gap Hausdorff? Is every special gap oriented?

# Special gaps.

## Basic facts

- notation: "Hausdorff" = "equivalent to Hausdorff";
- Hausdorff $\Longrightarrow$ oriented $\Longrightarrow$ special;
- If $(L_\alpha, R_\alpha)_\alpha$ is Hausdorff, then $(L_\alpha)_\alpha$ is Hausdorff;
- If $(L_\alpha, R_\alpha)_\alpha$ is oriented, then $(L_\alpha)_\alpha$ is special.

## Questions - Scheepers (1993)

Is every oriented gap Hausdorff? Is every special gap oriented?

# Special gaps.

## Basic facts

- notation: "Hausdorff" = "equivalent to Hausdorff";
- Hausdorff $\Longrightarrow$ oriented $\Longrightarrow$ special;
- If $(L_\alpha, R_\alpha)_\alpha$ is Hausdorff, then $(L_\alpha)_\alpha$ is Hausdorff;
- If $(L_\alpha, R_\alpha)_\alpha$ is oriented, then $(L_\alpha)_\alpha$ is special.

## Questions - Scheepers (1993)

Is every oriented gap Hausdorff? Is every special gap oriented?

# Special gaps.

**Questions - Scheepers (1993)**

Is every oriented gap Hausdorff? Is every special gap oriented?

**Theorem - Hirschorn (2008)**

Consistently, there is an oriented but non-Hausdorff gap.

**Theorem**

Consistently, there is a special gap which is not oriented.

# Special gaps.

**Questions - Scheepers (1993)**

Is every oriented gap Hausdorff? Is every special gap oriented?

**Theorem - Hirschorn (2008)**

Consistently, there is an oriented but non-Hausdorff gap.

**Theorem**

Consistently, there is a special gap which is not oriented.

# Special gaps.

**Questions - Scheepers (1993)**

Is every oriented gap Hausdorff? Is every special gap oriented?

**Theorem - Hirschorn (2008)**

Consistently, there is an oriented but non-Hausdorff gap.

**Theorem**

Consistently, there is a special gap which is not oriented.

# Special, non-oriented gap - proof.

- Let $(T_\alpha)_\alpha$ be a Suslin tower.
- (Spasojević, 1996) There is a $\sigma$-centered forcing $\mathbb{P}$ and a $\mathbb{P}$-name $(\dot{L}_\alpha)_\alpha$ for a tower such that

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is an oriented gap.}$$

- Consider $(T_\alpha, \dot{L}_\alpha)_\alpha$. It is still **special** in $V^{\mathbb{P}}$.
- $\Vdash_{\mathbb{P}}$ "$(T_\alpha)_\alpha$ is Suslin", since $\sigma$-centered forcings cannot destroy ccc.
- Therefore, $(T_\alpha, \dot{L}_\alpha)_\alpha$ is **not oriented**.

# Special, non-oriented gap - proof.

- Let $(T_\alpha)_\alpha$ be a Suslin tower.
- (Spasojević, 1996) There is a $\sigma$-centered forcing $\mathbb{P}$ and a $\mathbb{P}$-name $(\dot{L}_\alpha)_\alpha$ for a tower such that

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is an oriented gap.}$$

- Consider $(T_\alpha, \dot{L}_\alpha)_\alpha$. It is still **special** in $V^{\mathbb{P}}$.
- $\Vdash_{\mathbb{P}}$ "$(T_\alpha)_\alpha$ is Suslin", since $\sigma$-centered forcings cannot destroy ccc.
- Therefore, $(T_\alpha, \dot{L}_\alpha)_\alpha$ is **not oriented**.

# Special, non-oriented gap - proof.

- Let $(T_\alpha)_\alpha$ be a Suslin tower.
- (Spasojević, 1996) There is a $\sigma$-centered forcing $\mathbb{P}$ and a $\mathbb{P}$-name $(\dot{L}_\alpha)_\alpha$ for a tower such that

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is an oriented gap.}$$

- Consider $(T_\alpha, \dot{L}_\alpha)_\alpha$. It is still **special** in $V^{\mathbb{P}}$.
- $\Vdash_{\mathbb{P}}$ "$(T_\alpha)_\alpha$ is Suslin", since $\sigma$-centered forcings cannot destroy ccc.
- Therefore, $(T_\alpha, \dot{L}_\alpha)_\alpha$ is **not oriented**.

# Special, non-oriented gap - proof.

- Let $(T_\alpha)_\alpha$ be a Suslin tower.
- (Spasojević, 1996) There is a $\sigma$-centered forcing $\mathbb{P}$ and a $\mathbb{P}$-name $(\dot{L}_\alpha)_\alpha$ for a tower such that

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is an oriented gap.}$$

- Consider $(T_\alpha, \dot{L}_\alpha)_\alpha$. It is still **special** in $V^{\mathbb{P}}$.
- $\Vdash_{\mathbb{P}}$ "$(T_\alpha)_\alpha$ is Suslin", since $\sigma$-centered forcings cannot destroy ccc.
- Therefore, $(T_\alpha, \dot{L}_\alpha)_\alpha$ is **not oriented**.

# Special, non-oriented gap - proof.

- Let $(T_\alpha)_\alpha$ be a Suslin tower.
- (Spasojević, 1996) There is a $\sigma$-centered forcing $\mathbb{P}$ and a $\mathbb{P}$-name $(\dot{L}_\alpha)_\alpha$ for a tower such that

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is an oriented gap.}$$

- Consider $(T_\alpha, \dot{L}_\alpha)_\alpha$. It is still **special** in $V^{\mathbb{P}}$.
- $\Vdash_{\mathbb{P}}$ "$(T_\alpha)_\alpha$ is Suslin", since $\sigma$-centered forcings cannot destroy ccc.
- Therefore, $(T_\alpha, \dot{L}_\alpha)_\alpha$ is **not oriented**.

# A "universal" counterexample.

- A surprise:

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha)_\alpha \text{ is not a Hausdorff tower.}$$

- Therefore

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is oriented, not Hausdorff.}$$

- And

$$\Vdash_{\mathbb{P}} (T_\alpha, \dot{L}_\alpha)_\alpha \text{ is special, not oriented.}$$

# A "universal" counterexample.

- A surprise:

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha)_\alpha \text{ is not a Hausdorff tower.}$$

- Therefore

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is oriented, not Hausdorff.}$$

- And

$$\Vdash_{\mathbb{P}} (T_\alpha, \dot{L}_\alpha)_\alpha \text{ is special, not oriented.}$$

# A "universal" counterexample.

- A surprise:

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha)_\alpha \text{ is not a Hausdorff tower.}$$

- Therefore

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is oriented, not Hausdorff.}$$

- And

$$\Vdash_{\mathbb{P}} (T_\alpha, \dot{L}_\alpha)_\alpha \text{ is special, not oriented.}$$

# A "universal" counterexample.

- A surprise:

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha)_\alpha \text{ is not a Hausdorff tower.}$$

- Therefore

$$\Vdash_{\mathbb{P}} (\dot{L}_\alpha, T_\alpha)_\alpha \text{ is oriented, not Hausdorff.}$$

- And

$$\Vdash_{\mathbb{P}} (T_\alpha, \dot{L}_\alpha)_\alpha \text{ is special, not oriented.}$$

# Special non-Hausdorff tower.

## Corollary

Consistently, there is a special tower which is not Hausdorff.

## Theorem

- If $(T_\alpha)_\alpha$, $(T'_\alpha)_\alpha$ generates the same ideal, and $(T_\alpha)_\alpha$ is Hausdorff, then $(T'_\alpha)_\alpha$ is Hausdorff.
- Consistently, there is a special tower and a Suslin tower generating the same ideal.

# Special non-Hausdorff tower.

## Corollary

Consistently, there is a special tower which is not Hausdorff.

## Theorem

- If $(T_\alpha)_\alpha$, $(T'_\alpha)_\alpha$ generates the same ideal, and $(T_\alpha)_\alpha$ is Hausdorff, then $(T'_\alpha)_\alpha$ is Hausdorff.
- Consistently, there is a special tower and a Suslin tower generating the same ideal.

# Special non-Hausdorff tower.

## Corollary

Consistently, there is a special tower which is not Hausdorff.

## Theorem

- If $(T_\alpha)_\alpha$, $(T'_\alpha)_\alpha$ generates the same ideal, and $(T_\alpha)_\alpha$ is Hausdorff, then $(T'_\alpha)_\alpha$ is Hausdorff.
- Consistently, there is a special tower and a Suslin tower generating the same ideal.

# Acknowledgements: INFTY Network.

# Acknowledgements: Fields Institute.

Figure: CN tower