Universal homomorphisms

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Definition

The Urysohn space is the unique Polish metric space $\mathbb{U}$ satisfying the following condition:

(U) Given finite metric spaces $S \subseteq T$, given an isometric embedding $f\colon S \to \mathbb{U}$, given $\varepsilon > 0$, there exists an ε-isometric embedding $g\colon T \to \mathbb{U}$ such that $g \restriction S = f$.

Definition

The Gurarii space is the unique separable Banach space $\mathbb{G}$ satisfying the following condition:

(G) Given finite-dimensional spaces $S \subseteq T$, given a linear isometric embedding $f\colon S \to \mathbb{U}$, given $\varepsilon > 0$, there exists a linear ε-isometric embedding $g\colon T \to \mathbb{U}$ such that $g \restriction S = f$.

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Theorem

Let $\mathbb{U}$ denote the Urysohn space. There exists a non-expansive map $v \colon \mathbb{U} \to \mathbb{U}$ satisfying the following conditions:

1. *For every non-expansive map $f \colon X \to Y$ between separable metric spaces, there exist isometric embeddings $i \colon X \to \mathbb{U}$, $j \colon Y \to \mathbb{U}$ such that $v \circ i = j \circ f$.*
2. *Given isometries $g \colon A_0 \to A_1$, $h \colon B_0 \to B_1$ such that $A_0, A_1, B_0, B_1 \subseteq \mathbb{U}$ are finite and $h \circ v = v \circ g$, there exist bijective isometries $G \colon \mathbb{U} \to \mathbb{U}$ and $H \colon \mathbb{U} \to \mathbb{U}$ extending g and h, respectively, and such that $H \circ v = v \circ G$.*

Furthermore, the conditions above determine v up to an isometry.

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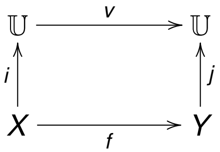

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Let X be a Polish metric space. Then there exist a Polish space $U(X) \supseteq X$ and a non-expansive map $u \colon U(X) \to X$ such that $u \restriction X = \mathrm{id}_X$ and the following conditions are satisfied:

1. *For every Polish metric space Y, for every non-expansive map $f \colon Y \to X$ there exists an isometric embedding $i \colon Y \to U(X)$ such that $f = u \circ i$.*
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Definition

A metric space $\langle X, d \rangle$ is finitely hyperconvex if for every family $B_0, \dots, B_{n-1}$ consisting of closed balls such that

$$\bigcap_{i<n} B_i = \emptyset$$

there exist $i < j < n$ such that $d(x_i, x_j) > r_i + r_j$, where $B_i = \overline{B}(x_i, r_i)$ and $B_j = \overline{B}(x_j, r_j)$.

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Given a Polish metric space X the following conditions are equivalent:

(a) *X is finitely hyperconvex.*

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Theorem

Given a Polish metric space X the following conditions are equivalent:

(a) *X is finitely hyperconvex.*

(b) *$U(X)$ is isometric to the Urysohn space $\mathbb{U}$.*

Corollary (K. 2011)

Given a Polish metric space X, the following properties are equivalent:

(a) *X is a non-expansive retract of the Urysohn space $\mathbb{U}$.*

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Let $u\colon U(X) \to X$ be as before. Then for every $p \in X$ the subspace $u^{-1}(p)$ is isometric to $\mathbb{U}$.

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Theorem (Garbulińska & K.)

Let $\mathbb{G}$ denote the Gurarii space. There exists a norm 1 linear operator $v\colon \mathbb{G} \to \mathbb{G}$ satisfying the following conditions:

1. *For every norm 1 linear operator $f\colon X \to Y$ between separable Banach spaces, there exist isometric embeddings $i\colon X \to \mathbb{G}$, $j\colon Y \to \mathbb{G}$ such that $v \circ i = j \circ f$.*
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Theorem (Wojtaszczyk 1972, Lusky 1977)

Given a separable Banach space X, the following conditions are equivalent:

(a) *X is an isometric L^1 predual.*

(b) *X is linearly isometric to a 1-complemented subspace of $\mathbb{G}$.*

(c) *There exists a norm 1 projection $P \colon \mathbb{G} \to \mathbb{G}$ such that $\operatorname{im} P$ is linearly isometric to X and $\ker P$ is linearly isometric to $\mathbb{G}$.*

Classical Fraïssé theory

The setup: Fraïssé class

- $\mathscr{F}$ is a class of finitely generated structures.
- Joint Embedding Property: Given $X, Y \in \mathscr{F}$, there is $Z \in \mathscr{F}$ such that both $X \hookrightarrow Z$ and $Y \hookrightarrow Z$.
- Amalgamation Property: Given embeddings $i \colon Z \hookrightarrow X$, $j \colon Z \hookrightarrow Y$ with $Z, X, Y \in \mathscr{F}$, there exists $W \in \mathscr{F}$ such that for some embeddings the diagram

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Fraïssé theorem

$$\sigma\mathscr{F} := \Big\{ \bigcup_{n\in\omega} X_n \colon \{X_n\}_{n\in\omega} \subseteq \mathscr{F} \text{ is a chain} \Big\}$$

Theorem

Let $\mathscr{F}$ be a countable Fraïssé class. Then there exists a unique, up to isomorphism, countable structure $U = \operatorname{Flim}\mathscr{F}$, satisfying the following conditions.

1. $U \in \sigma\mathscr{F}$.
2. *Given $\mathscr{F}$-structures $X \subseteq Y$, given an embedding $e\colon X \hookrightarrow U$, there exists an embedding $f\colon Y \hookrightarrow U$ such that $f \restriction X = e$.*
3. *Every $\mathscr{F}$-structure embeds into U.*

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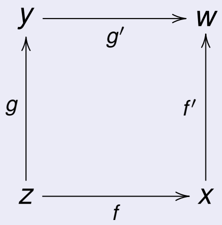
Main ingredient: pushouts

A pushout square



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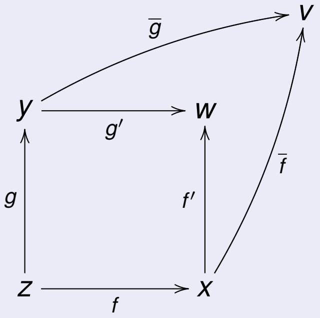

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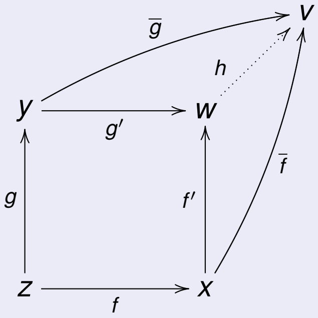

Mixed pushouts

Definition

Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories. We say that $\mathfrak{K}$ has mixed pushouts in $\mathfrak{L}$ if for every $\mathfrak{K}$-arrow $i \colon C \to A$, for every $\mathfrak{L}$-arrow $f \colon C \to B$, there exist a $\mathfrak{K}$-arrow $j \colon B \to W$ and an $\mathfrak{L}$-arrow $g \colon A \to W$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & W \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ C & \xrightarrow[f]{} & B \end{array}$$

commutes.

Universal homomorphisms, I

Theorem

Let $\mathfrak{M}$ be a countable Fraïssé class of finitely generated models, with the mixed pushout property. Let W denote the Fraïssé limit of $\mathfrak{M}$. Then there exists a unique (up to isomorphism) homomorphism $L \colon W \to W$ satisfying the following conditions.

(a) *For every $X, Y \in \sigma\mathfrak{M}$, for every homomorphism $F \colon X \to Y$ there exist embeddings $I_X \colon X \to W$ and $I_Y \colon Y \to W$ such that $L \circ I_X = I_Y \circ F$.*

(b) *Given finitely generated substructures x_0, x_1, y_0, y_1 of W such that $L[x_i] \subseteq y_i$ for $i < 2$, given isomorphisms $h_i \colon x_i \to y_i$ for $i < 2$ such that $L \circ h_0 = h_1 \circ L$, there exist automorphisms $H_i \colon W \to W$ extending h_i for $i < 2$, and such that $L \circ H_0 = H_1 \circ L$.*

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(b) *Given finitely generated substructures x_0, x_1, y_0, y_1 of W such that $L[x_i] \subseteq y_i$ for $i < 2$, given isomorphisms $h_i \colon x_i \to y_i$ for $i < 2$ such that $L \circ h_0 = h_1 \circ L$, there exist automorphisms $H_i \colon W \to W$ extending h_i for $i < 2$, and such that $L \circ H_0 = H_1 \circ L$.*

Universal homomorphisms, I

Theorem

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$$\begin{array}{ccc} W & \xrightarrow{L} & W \\ {\scriptstyle I_X}\uparrow & & \uparrow{\scriptstyle I_Y} \\ X & \xrightarrow[F]{} & Y \end{array}$$

Universal homomorphisms, II

Theorem (Pech & Pech 2012)

Let $\mathfrak{M}$ be as before and let $X \in \sigma\mathfrak{M}$. Then there exists $U(X) \in \sigma\mathfrak{M}$ such that $X \subseteq U(X)$, and there exists a homomorphism $u \colon U(X) \to X$ satisfying

1. *For every $Y \in \sigma\mathfrak{M}$ and for every homomorphism $f \colon Y \to X$ there is an embedding $i \colon Y \to U(X)$ such that $u \circ i = f$.*
2. *For every finitely generated substructures $S, T \subseteq U(X)$, for every isomorphism $h \colon S \to T$ such that $u \circ h = u$, there exists an isomorphism $H \colon U(X) \to U(X)$ satisfying $H \restriction S = h$ and $u \circ H = u$.*

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The key tool

Definition

A sequence

$$u_0 \longrightarrow u_1 \longrightarrow u_2 \longrightarrow \cdots$$

is Fraïssé

if for every n, for every $\mathfrak{K}$-arrow $f\colon u_n \to y$ there exist $m \geqslant n$ and a $\mathfrak{K}$-arrow $g\colon y \to u_m$ such that $g \circ f = u_n^m$.

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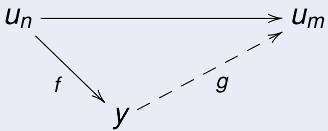

Relevant categories

Fix a class $\mathfrak{M}$ of finitely generated models. Let $\mathfrak{K}$ be the category whose objects are homomorphisms $f \colon X \to Y$, where $X, Y \in \mathfrak{M}$. A $\mathfrak{K}$-arrow from $f \colon X \to Y$ to $g \colon Z \to V$ is a pair of embeddings $\langle i, j \rangle$ satisfying $g \circ i = j \circ f$.

Claim

If $\mathfrak{M}$ is countable and has the mixed pushout property then $\mathfrak{K}$ has a Fraïssé sequence.

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Claim

If $\mathfrak{M}$ is countable and has the mixed pushout property then $\mathfrak{K}$ has a Fraïssé sequence.

Now fix $X \in \sigma\mathfrak{M}$. Let $\mathfrak{L}$ be the category whose objects are homomorphisms $f\colon A \to X$, where $A \in \mathfrak{M}$. An $\mathfrak{L}$-arrow from $f\colon A \to X$ to $g\colon B \to X$ is an embedding $i\colon A \to B$ such that $g \circ i = f$.

Claim

If $\mathfrak{M}$ is countable and has the mixed pushout property then $\mathfrak{L}$ has a Fraïssé sequence.

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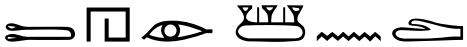

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