Universal homomorphisms

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Definition

The **Urysohn space** is the unique Polish metric space $U$ satisfying the following condition:

(U) Given finite metric spaces $S \subseteq T$, given an isometric embedding $f : S \to U$, given $\epsilon > 0$, there exists an $\epsilon$-isometric embedding $g : T \to U$ such that $g \upharpoonright S = f$.

Definition

The **Gurarii space** is the unique separable Banach space $G$ satisfying the following condition:

(G) Given finite-dimensional spaces $S \subseteq T$, given a linear isometric embedding $f : S \to U$, given $\epsilon > 0$, there exists a linear $\epsilon$-isometric embedding $g : T \to U$ such that $g \upharpoonright S = f$. 
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Theorem

Let $\mathbb{U}$ denote the Urysohn space. There exists a non-expansive map $v: \mathbb{U} \to \mathbb{U}$ satisfying the following conditions:

1. For every non-expansive map $f: X \to Y$ between separable metric spaces, there exist isometric embeddings $i: X \to \mathbb{U}$, $j: Y \to \mathbb{U}$ such that $v \circ i = j \circ f$.

2. Given isometries $g: A_0 \to A_1$, $h: B_0 \to B_1$ such that $A_0, A_1, B_0, B_1 \subseteq \mathbb{U}$ are finite and $h \circ v = v \circ g$, there exist bijective isometries $G: \mathbb{U} \to \mathbb{U}$ and $H: \mathbb{U} \to \mathbb{U}$ extending $g$ and $h$, respectively, and such that $H \circ v = v \circ G$.

Furthermore, the conditions above determine $v$ up to an isometry.
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Definition

A metric space \( \langle X, d \rangle \) is **finitely hyperconvex** if for every family \( B_0, \ldots, B_{n-1} \) consisting of closed balls such that
\[
\bigcap_{i<n} B_i = \emptyset
\]
there exist \( i < j < n \) such that \( d(x_i, x_j) > r_i + r_j \), where \( B_i = \overline{B}(x_i, r_i) \) and \( B_j = \overline{B}(x_j, r_j) \).

Theorem

Given a Polish metric space \( X \) the following conditions are equivalent:

(a) \( X \) is finitely hyperconvex.

(b) \( U(X) \) is isometric to the Urysohn space \( \mathbb{U} \).
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Theorem

Given a Polish metric space $X$ the following conditions are equivalent:

(a) $X$ is finitely hyperconvex.
(b) $U(X)$ is isometric to the Urysohn space $\mathbb{U}$. 
Corollary (K. 2011)

Given a Polish metric space $X$, the following properties are equivalent:

(a) $X$ is a non-expansive retract of the Urysohn space $\mathbb{U}$.
(b) $X$ is finitely hyperconvex.

Theorem

Let $u : U(X) \to X$ be as before. Then for every $p \in X$ the subspace $u^{-1}(p)$ is isometric to $\mathbb{U}$. 
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Theorem (Garbulińska & K.)

Let \( G \) denote the Gurarii space. There exists a norm 1 linear operator \( v : G \to G \) satisfying the following conditions:

1. For every norm 1 linear operator \( f : X \to Y \) between separable Banach spaces, there exist isometric embeddings \( i : X \to G \), \( j : Y \to G \) such that \( v \circ i = j \circ f \).

2. Given linear isometries \( g : A_0 \to A_1 \), \( h : B_0 \to B_1 \) such that \( A_0, A_1, B_0, B_1 \subseteq G \) are finite-dimensional spaces and \( h \circ v = v \circ g \), given \( \varepsilon > 0 \), there exist bijective isometries \( G : G \to G \) and \( H : G \to G \) extending \( g \) and \( h \), respectively, and such that \( \| H \circ v - v \circ G \| < \varepsilon \).

Furthermore, the conditions above determine \( v \) up to an isometry.
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Theorem (Wojtaszczyk 1972, Lusky 1977)

Given a separable Banach space $X$, the following conditions are equivalent:

(a) $X$ is an isometric $L^1$ predual.

(b) $X$ is linearly isometric to a 1-complemented subspace of $G$.

(c) There exists a norm 1 projection $P: G \to G$ such that $\text{im } P$ is linearly isometric to $X$ and $\text{ker } P$ is linearly isometric to $G$. 

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Classical Fraïssé theory

The setup: Fraïssé class

- $\mathcal{F}$ is a class of finitely generated structures.
- Joint Embedding Property: Given $X, Y \in \mathcal{F}$, there is $Z \in \mathcal{F}$ such that both $X \hookrightarrow Z$ and $Y \hookrightarrow Z$.
- Amalgamation Property: Given embeddings $i: Z \hookrightarrow X, j: Z \hookrightarrow Y$ with $Z, X, Y \in \mathcal{F}$, there exists $W \in \mathcal{F}$ such that for some embeddings the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow j & & \downarrow \\
Y & \xrightarrow{} & W
\end{array}
\]

commutes.
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Fraïssé theorem

\[ \sigma \mathcal{F} := \left\{ \bigcup_{n \in \omega} X_n : \{X_n\}_{n \in \omega} \subseteq \mathcal{F} \text{ is a chain} \right\} \]

Theorem

Let \( \mathcal{F} \) be a countable Fraïssé class. Then there exists a unique, up to isomorphism, countable structure \( U = \text{Flim } \mathcal{F} \), satisfying the following conditions.

1. \( U \in \sigma \mathcal{F} \).
2. Given \( \mathcal{F} \)-structures \( X \subseteq Y \), given an embedding \( e : X \hookrightarrow U \), there exists an embedding \( f : Y \hookrightarrow U \) such that \( f \upharpoonright X = e \).
3. Every \( \mathcal{F} \)-structure embeds into \( U \).
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Main ingredient: pushouts

A pushout square

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\begin{array}{ccc}
  y & \xrightarrow{g'} & w \\
  \uparrow^{g} & & \uparrow^{f'} \\
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  & V & \\
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\end{array}
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w \\
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\end{array}
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f \\
\end{array}
\]

\text{f} \quad \text{h} \quad \bar{f}
Mixed pushouts

Definition

Let $\mathcal{K} \subseteq \mathcal{L}$ be two categories. We say that $\mathcal{K}$ has mixed pushouts in $\mathcal{L}$ if for every $\mathcal{K}$-arrow $i: C \rightarrow A$, for every $\mathcal{L}$-arrow $f: C \rightarrow B$, there exist a $\mathcal{K}$-arrow $j: B \rightarrow W$ and an $\mathcal{L}$-arrow $g: A \rightarrow W$ such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{g} & W \\
\uparrow i & & \uparrow j \\
C & \xrightarrow{f} & B
\end{array}
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commutes.
Let $\mathcal{M}$ be a countable Fraïssé class of finitely generated models, with the mixed pushout property. Let $W$ denote the Fraïssé limit of $\mathcal{M}$. Then there exists a unique (up to isomorphism) homomorphism $L : W \to W$ satisfying the following conditions.

(a) For every $X, Y \in \sigma\mathcal{M}$, for every homomorphism $F : X \to Y$ there exist embeddings $I_X : X \to W$ and $I_Y : Y \to W$ such that $L \circ I_X = I_Y \circ F$.

(b) Given finitely generated substructures $x_0, x_1, y_0, y_1$ of $W$ such that $L[x_i] \subseteq y_i$ for $i < 2$, given isomorphisms $h_i : x_i \to y_i$ for $i < 2$ such that $L \circ h_0 = h_1 \circ L$, there exist automorphisms $H_i : W \to W$ extending $h_i$ for $i < 2$, and such that $L \circ H_0 = H_1 \circ L$. 
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(b) Given finitely generated substructures $x_0, x_1, y_0, y_1$ of $W$ such that $L[x_i] \subseteq y_i$ for $i < 2$, given isomorphisms $h_i: x_i \to y_i$ for $i < 2$ such that $L \circ h_0 = h_1 \circ L$, there exist automorphisms $H_i: W \to W$ extending $h_i$ for $i < 2$, and such that $L \circ H_0 = H_1 \circ L$. 
Theorem

Let $\mathcal{M}$ be a countable Fraïssé class of finitely generated models, with the mixed pushout property. Let $W$ denote the Fraïssé limit of $\mathcal{M}$. Then there exists a unique (up to isomorphism) homomorphism $L: W \to W$ satisfying the following conditions.

(a) For every $X, Y \in \sigma \mathcal{M}$, for every homomorphism $F: X \to Y$ there exist embeddings $I_X: X \to W$ and $I_Y: Y \to W$ such that $L \circ I_X = I_Y \circ F$.

(b) Given finitely generated substructures $x_0, x_1, y_0, y_1$ of $W$ such that $L[x_i] \subseteq y_i$ for $i < 2$, given isomorphisms $h_i: x_i \to y_i$ for $i < 2$ such that $L \circ h_0 = h_1 \circ L$, there exist automorphisms $H_i: W \to W$ extending $h_i$ for $i < 2$, and such that $L \circ H_0 = H_1 \circ L$. 
Theorem (Pech & Pech 2012)

Let $\mathcal{M}$ be as before and let $X \in \sigma \mathcal{M}$. Then there exists $U(X) \in \sigma \mathcal{M}$ such that $X \subseteq U(X)$, and there exists a homomorphism $u : U(X) \rightarrow X$ satisfying

1. For every $Y \in \sigma \mathcal{M}$ and for every homomorphism $f : Y \rightarrow X$ there is an embedding $i : Y \rightarrow U(X)$ such that $u \circ i = f$.

2. For every finitely generated substructures $S, T \subseteq U(X)$, for every isomorphism $h : S \rightarrow T$ such that $u \circ h = u$, there exists an isomorphism $H : U(X) \rightarrow U(X)$ satisfying $H \upharpoonright S = h$ and $u \circ H = u$. 
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The key tool

Definition

A sequence

\[ u_0 \rightarrow u_1 \rightarrow u_2 \rightarrow \cdots \]

is Fraïssé

if for every \( n \), for every \( K \)-arrow \( f: u_n \rightarrow y \) there exist \( m \geq n \) and a \( K \)-arrow \( g: y \rightarrow u_m \) such that \( g \circ f = u_m^m \).

\[ u_n \rightarrow u_m \]

\[ y \]

\[ f \]

\[ g \]
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\[ u_n \quad \xrightarrow{f} \quad y \quad \xleftarrow{g} \quad u_m \]
Fix a class $\mathcal{M}$ of finitely generated models. Let $\mathcal{K}$ be the category whose objects are homomorphisms $f: X \to Y$, where $X, Y \in \mathcal{M}$. A $\mathcal{K}$-arrow from $f: X \to Y$ to $g: Z \to V$ is a pair of embeddings $\langle i, j \rangle$ satisfying $g \circ i = j \circ f$.

Claim

If $\mathcal{M}$ is countable and has the mixed pushout property then $\mathcal{K}$ has a Fraïssé sequence.
Fix a class \( \mathcal{M} \) of finitely generated models. Let \( \mathcal{K} \) be the category whose objects are homomorphisms \( f: X \to Y \), where \( X, Y \in \mathcal{M} \). A \( \mathcal{K} \)-arrow from \( f: X \to Y \) to \( g: Z \to V \) is a pair of embeddings \( \langle i, j \rangle \) satisfying \( g \circ i = j \circ f \).

**Claim**

*If \( \mathcal{M} \) is countable and has the mixed pushout property then \( \mathcal{K} \) has a Fraïssé sequence.*
Now fix $X \in \sigma \mathcal{M}$. Let $\mathcal{L}$ be the category whose objects are homomorphisms $f : A \to X$, where $A \in \mathcal{M}$. An $\mathcal{L}$-arrow from $f : A \to X$ to $g : B \to X$ is an embedding $i : A \to B$ such that $g \circ i = f$.

Claim

*If $\mathcal{M}$ is countable and has the mixed pushout property then $\mathcal{L}$ has a Fraïssé sequence.*
References


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