Dimension(s) of compact $F$-spaces

Quidquid latine dictum sit, altum videtur

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Outline

1. History
2. Can we generalize?
3. Finite-to-one maps
4. Questions
5. Sources
Hurewicz’ theorem

Theorem

Let $X$ be separable and metrizable and $n \in \mathbb{N}$. Let $X$ be separable and metrizable and $n \in \mathbb{N}$. Then the dimension of $X$ is at most $n$ if and only if there are a zero-dimensional, separable and metrizable space $Y$ and a closed continuous surjection $f: Y \to X$ such that $\|f^{-1}(x)\| \leq n + 1$ for all $x \in X$. 
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One direction uses the large inductive dimension.

**Theorem**

*If $Y$ is normal and strongly zero-dimensional and $f : Y \to X$ is closed, continuous and onto with $|f^{-1}(x)| \leq n + 1$ for all $x \in X$ then $\text{Ind} \ X \leq n$.***
Proof.

By induction (of course).
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Given disjoint closed sets $A$ and $B$ in $X$ find a closed set $Z$ in $Y$ such that $f[Z]$ is a partition between and $f \upharpoonright Z$ has fibers of size at most $n$. 
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Given disjoint closed sets $A$ and $B$ in $X$ find a closed set $Z$ in $Y$ such that $f[Z]$ is a partition between and $f \upharpoonright Z$ has fibers of size at most $n$.
The speaker draws an instructive picture . . .
The other direction uses the covering dimension dimension. \(\dim X \leq n\) iff for every open cover \(\mathcal{U}\) of \(X\) of cardinality \(n + 2\) there is an open refinement \(\mathcal{V} = \{V_U : U \in \mathcal{U}\}\) with \(\bigcap\{\text{cl } V : V \in \mathcal{V}\} = \emptyset\). Refinement: \(V_U \subseteq U\) for all \(U\) and \(\bigcup \mathcal{V} = X\).
Theorem

If $X$ is compact and metrizable with $\dim X \leq n$ then there are a zero-dimensional, compact and metrizable space $Y$ and a continuous surjection $F : Y \to X$ with $|f^{-1}(x)| \leq n + 1$ for all $x \in X$. 
Proof.

Idea of proof: make finite closed covers of order at most $n + 1$; give these the discrete topology; take their product and let $Y$ be a suitable subspace of that product.
The reason we have an equivalence is the fundamental fact from dimension theory that $\dim X = \text{ind } X = \text{Ind } X$ for all separable and metrizable $X$.
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And the compactification theorem: a separable and metrizable space has a metric compactification with the same dimension(s).
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F-spaces of weight $c$

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Or, somewhat more topological: disjoint cozero sets are completely separated.

Or, for normal spaces: disjoint cozero sets have disjoint closures.
For every compact $F$-space, $X$, of weight $c$ we have

$$\dim X = \text{ind } X = \text{Ind } X$$
Equality of dimensions

**Theorem (CH)**

*For every compact F-space, X, of weight \( c \) we have*

\[
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\]

**Proof**

The inequalities \( \dim X \leq \text{ind } X \leq \text{Ind } X \) hold for every compact space.
Proof, continued

Proof

The interesting part is the proof of $\text{Ind } X \leq \dim X$. 

Given disjoint closed sets $A$ and $B$ we build a partition, $L$, between them with $\dim L \leq \dim X - 1$. How: we have $\aleph_1$ many potential basic open covers of $L$ of size $\dim X + 1$; enumerate them: $\langle U_\alpha : \alpha < \omega_1 \rangle$. Build increasing sequences $\langle C_\alpha : \alpha < \omega_1 \rangle$ and $\langle D_\alpha : \alpha < \omega_1 \rangle$ of cozero sets, with $C_\alpha \cap D_\alpha = \emptyset$ for all $\alpha$. 

K. P. Hart
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At stage \( \alpha \), check if \( C_\alpha \cup D_\alpha \cup \bigcup U_\alpha = X \).
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At stage $\alpha$, check if $C_\alpha \cup D_\alpha \cup \bigcup U_\alpha = X$. In that case take a refinement $\{O\} \cup \mathcal{V}_\alpha$ of $\{C_\alpha \cup D_\alpha\} \cup U_\alpha$ whose closures have empty intersection.
Proof, continued

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In that case take a refinement \( \{O\} \cup V_\alpha \) of \( \{C_\alpha \cup D_\alpha\} \cup U_\alpha \) whose closures have empty intersection.
Take \( C_{\alpha+1} \) and \( D_{\alpha+1} \) such that \( C_\alpha \cup \bigcap_{U \in U_\alpha} \text{cl} \ V_U \subseteq C_{\alpha+1} \) and \( D_\alpha \subseteq D_{\alpha+1} \).
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Apart from some technicalities this works.
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A very general theorem

**Theorem (CH)**

Let $X$ be a compact $F$-space of weight $\mathfrak{c}$. Then $X$ has a base $\{B_\alpha : \alpha < \omega_1\}$ with the following property:
A very general theorem

Theorem (CH)

Let $X$ be a compact $F$-space of weight $c$. Then $X$ has a base \[ \{ B_\alpha : \alpha < \omega_1 \} \] with the following property: whenever $F$ is a finite subset of $\omega_1$ the intersection

\[ \bigcap_{\alpha \in F} \text{Fr } B_\alpha \]

has dimension at most $\dim \text{Fr } B_{\min F} - |F| + 1$. 

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Dimension(s) of compact $F$-spaces
It uses a simultaneous version of the proof of $\text{Ind} \, X \leq \dim X$. 
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$$\dim(L \cap D) \leq \dim D - 1$$

for countably many closed sets $D$ at once.
It uses a simultaneous version of the proof of \( \text{Ind } X \leq \dim X \).

In the separable metric case one can build a partition, \( L \), such that

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\dim(L \cap D) \leq \dim D - 1
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for countably many closed sets \( D \) at once.

In the case of a compact \( F \)-space of weight \( \mathfrak{c} \), assuming CH, you can do this in one go for \( \aleph_1 \) many closed sets.
A special case

**Theorem (CH)**

Let $X$ be a compact $F$-space of weight $\mathfrak{c}$ and dimension $n$. Then $X$ has a base $\{B_\alpha : \alpha < \omega_1\}$ with the following property:

$$\bigcap_{\alpha \in F} \text{Fr} B_\alpha = \emptyset$$

whenever $F$ is a subset of $\omega_1$ with $n + 1$ elements.
A special case

Theorem (CH)

Let $X$ be a compact $F$-space of weight $\mathfrak{c}$ and dimension $n$. Then $X$ has a base $\{B_\alpha : \alpha < \omega_1\}$ with the following property:

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whenever $F$ is a subset of $\omega_1$ with $n + 1$ elements.
A finite-to-one map

We may assume our base consists of regular open sets \((B_\alpha = \text{int cl } B_\alpha)\).
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Take the Boolean subalgebra, \(B\), of RO(\(X\)) generated by our base.
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\( B_\alpha = \text{int} \text{ cl } B_\alpha \).
Take the Boolean subalgebra, \( B \), of RO(\( X \)) generated by our base. 
Then the natural map from the Stone space of \( B \) onto \( X \) is (at most) \( 2^n \)-to-one.
Bummer! $2^n > n + 1$ (when $n \geq 2$).
A finite-to-one map

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We have another proof, with the same result: $2^n$. 
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An example

The first question that should occur to everyone has an answer:

There is a compact $F$-space of weight $c+$ with non-coinciding dimensions (my student Jan van Mill). This parallels the 'classic' case: there are compact spaces of weight $\aleph_1$ with non-coinciding dimensions.
The first question that should occur to everyone has an answer: There is a compact $F$-space of weight $\mathfrak{c}^+$ with non-coinciding dimensions (my student Jan van Mill).
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There is a compact $F$-space of weight $\mathfrak{c}^+$ with non-coinciding dimensions (my student Jan van Mill).
This parallels the ‘classic’ case: there are compact spaces of weight $\aleph_1$ with non-coinciding dimensions.
What if CH fails?

The second question that should occur to everyone has no answer (yet).
What if CH fails?

The second question that should occur to everyone has no answer (yet).
One possibility: there are many compact spaces of weight \( c \) with non-coinciding dimensions.
What if CH fails?

Take such a space, $X$, for example with $\dim X = 1$ and $\text{ind } X = \text{Ind } X = 2$. 

Consider $Y = \omega \times X$ and $Y^* = \beta Y \setminus Y$. By our first result we have $\dim Y^* = \text{ind } Y^* = \text{Ind } Y^*$ if CH holds.
What if CH fails?

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What if CH fails?

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Consider $Y = \omega \times X$ and $Y^* = \beta Y \setminus Y$.
By our first result we have $\dim Y^* = \text{ind } Y^* = \text{Ind } Y^*$ if CH holds.
Last year’s tutorial: $\dim C = \dim X = 1$ for every component $C$ of $Y^*$. 
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Also \( \dim Y^* \leq \dim \beta Y = 1 \), so \( \dim Y^* = 1 \).
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Hence(!): \(\text{Ind } Y^* = 1 < 2 = \text{Ind } \beta Y\) (if CH).
What can be said if CH fails? In particular models where CH fails.
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What if CH fails?

What can be said if CH fails? In particular models where CH fails. Could it be that $\text{Ind } Y^* = 2$ in some such model? There are many $X$ to play with.
What with $2^n$?

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The third question on everyone’s lips: can $2^n$ be brought down to $n + 1$?

(As it should be.) We have no idea.
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Light reading

Website: fa.its.tudelft.nl/~hart

- **K. P. Hart, J. van Mill,**

- **J. van Mill,**
  *A compact F-space with noncoinciding dimensions,* Topology and its Applications **159** (2012), 1625–1633.
Let us thank the organizers