Remarks on Q-points and rapid ultrafilters

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Q-points and rapid ultrafilters

Definition.

A free ultrafilter $\mathcal{U}$ is called a Q-point if for every $\{Q_i : i \in \omega\}$, a partition of ω into finite sets, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap Q_i| \leq 1$.

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A free ultrafilter $\mathcal{U}$ is called rapid if for every $\{Q_i : i \in \omega\}$, a partition of ω into finite sets, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap Q_i| \leq i$.

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A free ultrafilter $\mathcal{U}$ is called a *Q-point* if for every $\{Q_i : i \in \omega\}$, a partition of ω into finite sets, there exists $U \in \mathcal{U}$ such that $(\forall i \in \omega)\ |U \cap Q_i| \leq 1$.

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Alternative definition of an rapid ultrafilter:

A free ultrafilter $\mathcal{U}$ is called rapid if the enumeration functions of its sets form a dominating family in $(\omega^\omega, \leq^*)$.

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In every model where Q-points are known not to exist, rapid ultrafilters do not exist either.

Generic existence

Definition (Canjar).

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Theorem (Canjar).

The following are equivalent:

- $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$,
- Q-points exist generically,
- Rapid ultrafilters exist generically.

Generic existence – questions

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Question 1. Which equality of cardinal invariants describes the generic existence of Q-points (rapid ultrafilters) if we consider the modified definition?

Question 2. Are general existence of Q-points and general existence of rapid ultrafilters equivalent also in this new definition?

Product of ultrafilters

Definition.

Let $\mathcal{U}$ and $\mathcal{V}$, $n \in \omega$, be ultrafilters on ω.
The product of ultrafilters $\mathcal{U}$ and $\mathcal{V}$, denoted by $\mathcal{U} \times \mathcal{V}$, is an ultrafilter on $\omega \times \omega$ defined by $A \in \mathcal{U} \times \mathcal{V}$ if and only if $\{n : \{m : \langle n, m\rangle \in A\} \in \mathcal{V}\} \in \mathcal{U}$.

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It is known that $\mathcal{U} \times \mathcal{V}$ is never a Q-point.

Theorem (Miller).

$\mathcal{U} \times \mathcal{V}$ is a rapid ultrafilter if and only if $\mathcal{V}$ is rapid.

Summable ideals

Definition.

Given a function $g : \omega \to [0, \infty)$ such that $\sum_{n \in \omega} g(n) = \infty$ then the family

$$\mathcal{I}_g = \{A \subseteq \omega : \sum_{a \in A} g(a) < +\infty\}$$

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A summable ideal is tall if and only if $\lim_{n \to \infty} g(n) = 0$.

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

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One can add two more equivalent conditions:

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- $(\forall f : \omega \to \omega$ finite-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$

$\mathcal{I}_g$-ultrafilters

Definition.

An ultrafilter $\mathcal{U} \in \omega^*$ is called an $\mathcal{I}_g$-ultrafilter if for every $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

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- If $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{I}_g$-ultrafilters then the ultrafilter product $\mathcal{U} \times \mathcal{V}$ is also an $\mathcal{I}_g$-ultrafilter.

$\mathcal{I}_g$-ultrafilters and Q-points

Theorem 4.

($\mathrm{MA}_{\mathrm{ctble}}$) For every tall ideal $\mathcal{I}$ there is a Q-point which is not an $\mathcal{I}$-ultrafilter.

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Theorem 5.

(MA_{ctble}) Assume $\{\mathcal{I}_\alpha : \alpha < \mathfrak{c}\}$ is a family of tall ideals. There is a Q-point which is not an $\mathcal{I}_\alpha$-ultrafilter for any $\alpha < \mathfrak{c}$.

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Corollary 6.

($\mathsf{MA_{ctble}}$) There is a Q-point which is not an $\mathcal{I}_g$-ultrafilter for any summable ideal $\mathcal{I}_g$.

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Theorem 7.

($\mathrm{MA}_{\mathrm{ctble}}$) There exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ and $\mathcal{U}$ is not a Q-point.

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Theorem 7.

($\mathrm{MA}_{\mathrm{ctble}}$) There exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ and $\mathcal{U}$ is not a Q-point.

The idea of the proof:

1. Take $\mathcal{V}$ which is $\mathcal{I}_g$-ultrafilter for every $\mathcal{I}_g$.

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The idea of the proof:

1. Take $\mathcal{V}$ which is $\mathcal{I}_g$-ultrafilter for every $\mathcal{I}_g$.
2. Put $\mathcal{U} = \mathcal{V} \times \mathcal{V}$ where $\mathcal{V}$.

$\mathcal{I}_g$-ultrafilters and rapid ultrafilters

Corollary 6.

(MA_{ctble}) There is a Q-point which is not an $\mathcal{I}_g$-ultrafilter for any summable ideal $\mathcal{I}_g$.

Corollary 8.

(MA_{ctble}) There is a rapid ultrafilter which is not an $\mathcal{I}_g$-ultrafilter for any summable ideal $\mathcal{I}_g$.

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Corollary 8.

(MA_{ctble}) There is a rapid ultrafilter which is not an $\mathcal{I}_g$-ultrafilter for any summable ideal $\mathcal{I}_g$.

If $\mathcal{U} \in \omega^*$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ then $\mathcal{U}$ is a rapid ultrafilter.

$\mathcal{I}_g$-ultrafilters and rapid ultrafilters

Theorem 9.

($\mathrm{MA}_{\mathrm{ctble}}$) There is an $\mathcal{I}_{\frac{1}{n}}$-ultrafilter which is not a rapid ultrafilter.

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Theorem 10.

(CH) For every tall summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_g$-ultrafilter which is not rapid.

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Definition.

A family $\mathcal{A} = \{A_{\alpha,n} : \alpha \in I, n \in \omega\} \subseteq \mathcal{P}(\omega)$ is called independent with respect to a filter base $\mathcal{F}$ if $\{A_{\alpha,n} : n \in \omega\}$ is a partition of ω into infinite sets for every $\alpha \in I$ and $(\forall B \in \mathcal{F})$ $(\forall M \in [I]^{<\omega})$ $(\forall f : M \to \omega)$ $|B \cap \bigcap_{\beta \in M} A_{\beta,f(\beta)}| = \omega$.

Proof of Theorem 5

Outline of the construction

1. List all partitions of ω into finite sets as $\{\mathcal{Q}_\alpha : \alpha < \mathfrak{c}\}$.

2. For $\alpha < \mathfrak{c}$ construct filter bases $\mathcal{F}_\alpha$ and families $\mathcal{A}_\alpha = \{A_{\beta,n} : \beta \leq \alpha, n \in \omega\}$ such that for every $\alpha < \mathfrak{c}$ the following hold:
   - (i) $\mathcal{F}_0$ is the Fréchet filter, $\mathcal{A}_0$ is partition of ω into infinite sets
   - (ii) $\mathcal{F}_\alpha \supseteq \mathcal{F}_\beta$, $\mathcal{A}_\alpha \supseteq \mathcal{A}_\beta$ whenever $\alpha \geq \beta$
   - (iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$, $\mathcal{A}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha$ for γ limit
   - (iv) $(\forall\alpha)$ $|\mathcal{F}_\alpha| \leq |\alpha| \cdot \omega$ and $|\mathcal{A}_\alpha| \leq |\alpha| \cdot \omega$
   - (v) $(\forall\alpha)$ $\mathcal{A}_\alpha$ is independent with respect to $\mathcal{F}_\alpha$
   - (vi) $(\forall\alpha)$ $(\exists B \in \mathcal{F}_{\alpha+1})$ $(\forall Q \in \mathcal{Q}_\alpha)$ $|B \cap Q| \leq 1$

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   (vi) $(\forall\alpha)\ (\exists B \in \mathcal{F}_{\alpha+1})\ (\forall Q \in \mathcal{Q}_\alpha)\ |B \cap Q| \leq 1$

3. Complete the induction step using two lemmas:

Proof of Theorem 5

Induction step

Lemma 5a.

(MA_{ctble}) Assume $\mathcal{F}$ is a filter base on ω with $|\mathcal{F}| < \mathfrak{c}$ and $\mathcal{A} = \{A_{\beta,n} : \beta \leq \alpha, n \in \omega\}$, $\alpha < \mathfrak{c}$ is independent w. r. t. $\mathcal{F}$. Then there exists a partition of ω into infinite sets $\{A_{\alpha+1,n} : n \in \omega\}$ such that $\mathcal{A}' = \mathcal{A} \cup \{A_{\alpha+1,n} : n \in \omega\}$ is independent with respect to $\mathcal{F}$.

Lemma 5b.

(MA_{ctble}) Let $\mathcal{F}$ be a filter base on ω with $|\mathcal{F}| < \mathfrak{c}$, $\mathcal{A} = \{A_{\beta,n} : \beta \leq \alpha, n \in \omega\}$, $\alpha < \mathfrak{c}$ an independent family w. r. t. to $\mathcal{F}$ and $\mathcal{Q} = \{Q_i : i \in \omega\}$ a partition of ω into finite sets. Then there exists $C \subseteq \omega$ such that $|C \cap Q| \leq 1$ for every $Q \in \mathcal{Q}$ and $\mathcal{A}$ is independent with respect to the filter base $\mathcal{F}'$ generated by $\mathcal{F}$ and C.

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Theorem 10a.

(CH) For arbitrary tall summable ideals $\mathcal{I}_g$ and $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$ there is an $\mathcal{I}_g$-ultrafilter $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I}_h = \emptyset$.

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Proposition 10b.

For every tall summable ideal $\mathcal{I}_g$ there is a tall summable ideal $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$.

Possible extension and its limits

Is it possible that an ultrafilter is an $\mathcal{I}_g$-ultrafilter for “many" tall summable ideals simultaneously and still not a rapid ultrafilter?

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Proposition 11.

There is a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \mathfrak{d}$ and an ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$.

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Proposition 12.

(CH) If $\mathcal{D}$ is a countable family of tall summable ideals then there is an ultrafilter $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter for every $\mathcal{I} \in \mathcal{D}$, but $\mathcal{U}$ is not a rapid ultrafilter.

Questions

Let $\mathcal{D}$ be a family of tall summable ideals.

Question 3.

What is the minimal size of the family $\mathcal{D}$ if rapid ultrafilters can be characterized as those ultrafilters on ω which have a nonempty intersection with all the ideals in the family $\mathcal{D}$?

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Let $\mathcal{D}$ be a family of tall summable ideals.

Question 3.

What is the minimal size of the family $\mathcal{D}$ if rapid ultrafilters can be characterized as those ultrafilters on ω which have a nonempty intersection with all the ideals in the family $\mathcal{D}$?

Question 4.

Is it true that whenever the cardinality of $\mathcal{D}$ is less than $\mathfrak{d}$ then there exist an ultrafilter on the natural numbers which is an $\mathcal{I}_g$-ultrafilter for every $\mathcal{I}_g \in \mathcal{D}$, but not a rapid ultrafilter?

References

R. M. Canjar, On the generic existence of special ultrafilters, *Proc. Amer. Math. Soc.* **110** (1990), 233 – 241.

J. Flašková, $\mathcal{I}$-ultrafilters and summable ideals, in: *Proceedings of the 10th Asian Logic Conference*, Kobe 2008.

A. Miller, There are no *Q*-points in Laver's model for the Borel conjecture, *Proc. Amer. Math. Soc.* **78** (1980), 498 – 502.

P. Vojtáš, On ω^* and absolutely divergent series, *Topology Proceedings* **19** (1994), 335 – 348.