A consistent solution of the Horn Tarski problem

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- P is an ordered set.
- **σ-finite cc**; there is a fragmentation

$$P = \bigcup_{n \in \omega} P_n$$

  such that there are only finitely many disjoint elements in each P_n.
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Measure and fragmentation properties

- A Boolean algebra carries a **strictly positive measure** μ if for any $a \cap b = 0$
$$\mu(a \vee b) = \mu(a) + \mu(b)$$
and
$$\mu(a) = 0 \text{ iff } a = \mathbf{0}.$$
- Fact: Any Boolean algebra carrying a strictly positive measure is σ-bounded cc.
- Witness: $P_n = \{a \ : \ \mu(a) > 1/n\}$.

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  and for each disjoint sequence $\langle a_n : n \in \omega \rangle \in B^\omega$, $\lim_{n\to\infty} \mu(a_n) = 0$ and

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- For a topological Hausdorff space X, let the Todorcevic ordering be

$$\mathbb{T}(X) = \{F \subseteq X \;:\; F \text{ is compact} \quad \& \quad |F^d| < \omega\}$$

where $F_1 \leq F_2$ if $F_1 \supseteq F_2$ and $F_1^d \cap F_2 = F_2^d$.

- $(S, \leq_S)$ Suslin tree
- $s \sim t$ iff $\forall r \in S : r <_S s \leftrightarrow r <_S t$
- every equivalence class of $\sim$: $\preceq$ ordering of type ω^*
- lexicographical order $\leq$ on S by $s < t$ if either $s <_S t$ or $s \not\leq_S t$ and there are $s' \leq_S s$ and $t' \leq_S t$ such that $s' \sim t'$ and $s' \prec t'$
- the interval topology $\tau_{\leq}$ on $(S, \leq)$

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- The ordering $\mathbb{T}(S, \tau_{\leq})$ is is σ-finite cc but not σ-bounded cc.

- **$\mathcal{P}$ is not σ-bounded cc.**
- notation: For any $s \in S$ choose in the ordering $\leq$ increasing $l(s,k)$ and decreasing $r(s,k)$ for $k < \omega$ such that $\sup\{l(s,k) : k < \omega\} = s = \inf\{r(s,k) : k < \omega\}$
- by contradiction: $\mathcal{P} = \bigcup_{n \in \omega} P_n$ such that there are at most n pairwise disjoint elements in P_n
- define $f_n : S \longrightarrow n+1$, such that $f_n(s)$ is the maximal length of an antichain which is a subset of the set $P_n(s) = \{F \in P_n : \exists t \in F^d (t \geq_S s)\}$
- f_n decreasing with respect to $\leq_S$
- for any $s \in S$ there is an $s' \geq_S s$ such that $f_n(s') = f_n(t)$ for all $t \geq_S s'$
- S Suslin $\rightarrow \exists s \in S$ ($f_n(s) = f_n(t)$ for all $t \geq_S s$ and all $n < \omega$)
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- $\{s_{n,i}\}_{n<\omega, i<f(n)}$ converges to s (if not finite)
- $F = \{s_{n,i}\}_{n<\omega, i<f(n)} \cup \{r(s,n)\}_{n<\omega} \cup \{s\} \in \mathcal{P}$
- F is orthogonal with all $F_{n,i}$ for $n < \omega$ and $i < f(n)$
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- $f_n(s) \geq f(n) + 1$, a contradiction

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- put $I(s,k) = \big(l(s,k), r(s,k)\big)$, the open interval with respect to the ordering $\leq$
- for any $F \in \mathcal{P}$ fix a $k(F) < \omega$ such that the $I(s,k(F))$ are mutually disjoint for $s \in F^d$
- $F = \dot{\bigcup}\{I(s,k(F)) \cap F \ : \ s \in F^d\} \ \dot{\cup} \ R(F)$, where $I(s,k(F)) \cap F$ is a converging sequence with limit s and $R(F)$ is a finite set of isolated points

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- $P_{k,n,m} = \{F \in \mathcal{P} \quad : \quad k(F) = k \;\&\; |F^d| = n \;\&\; |R(F)| = m\}$
- $\mathcal{P} = \bigcup_{k,n,m<\omega} P_{k,n,m}$
- all $P_{k,n,m}$ are finite-cc:
- by contradiction: $\mathcal{A} = \{F_i\}_{i<\omega} \subset P_{\bar{k},\bar{n},\bar{m}}$ is an infinite antichain for some fixed $\bar{k}, \bar{n}, \bar{m}$

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$\mathcal{P}$ IS σ-FINITE CC

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- Let $(F_i)^d = \{s_i^n\}_{n<\bar{n}}$ and $R(F_i) = \{r_i^m\}_{m<\bar{m}}$ be increasingly enumerated and put $F_i^n = F \cap I(s_i^n, \bar{k}) \setminus \{s_i^n\}$.
- $F_i \setminus (F_i)^d = \bigcup_{n<\bar{n}} F_i^n \cup \{r_i^m\}_{m<\bar{m}}$ is the set of isolated points of F_i
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We say that $\{i, j\} \in [\omega]^2$, $i < j$, has colour

$$\begin{array}{cl} (1, n, n', l) & \text{if } s_i^n \in F_j^{n'} \ \& \ s_i^n < s_j^{n'} \\ (1, n, n', r) & \text{if } s_i^n \in F_j^{n'} \ \& \ s_i^n > s_j^{n'} \\ (2, n, m) & \text{if } s_i^n = r_j^m \\ (3, n, n') & \text{if } s_j^n \in F_i^{n'} \\ (4, n, m) & \text{if } s_j^n = r_i^m \end{array}$$

for $n, n' < \bar{n}$ and $m < \bar{m}$.

- for any $\{i, j\} \in [\omega]^2$ there is a point which is isolated in F_i and not isolated in F_j or vice versa
- any pair $\{i, j\}$ obtains at least one colour
- Ramsey's theorem: $\mathcal{A}' \subseteq \mathcal{A}$ infinite homogeneous in one colour
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Outlook

- Assuming Martin‘s axiom, the above constructed example for an Aronshajn tree is even σ-bounded cc.
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