A short introduction to mereology

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Outline

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First steps

Mereology as a theory of part of relation

- the creator of mereology: polish logician and mathematician Stanisław Leśniewski (1886-1939)
- mereology can be characterized as a theory of *collective sets* in opposition to the Cantorian notion of *set*
- collective sets can be defined by means of *part of* relation, therefore mereology can be described as a theory of this relation
- *meros* means *part* in Greek.

Outline

Nominalism

Cantorian sets do not exists

- Leśniewski on Cantorian sets: *I can feel in them smell of mythical objects from rich gallery of figments of the imagination*.
- nothing like the empty set can exists
- his ontological stance admitted only concrete (spatio-temporal) objects
- series of papers titled *On foundations of mathematics*
- Leśniewski's mereology and its contemporary form.

Outline

Sets as collections and sets as aggregates

Sets can be understood in two ways:

- as collections, as in case of the set of all triangles in Euclidean space, the set of all natural numbers and so on;
- as aggregates (fusions, conglomerates), as such a brick wall can be seen as a fusion of bricks that were used to raise it; the fusion of all triangles in Euclidean space would be the whole space exactly.

The former notion is of course Cantorian, and common for contemporary set theory. The latter is mereological in nature.

Outline

Formalization

Basic properties of part of relation

- irreflexivity - nothing is a part of itself
- asymmetry - if x is a part of y, then y is not a part of x (it entails irreflexivity)
- transitivity - if x is a part of y and y is a part of z, then x is a part of z
- objections to transitivity - different notions of *parthood* (functional, spatio-temporal)

Formalization

Assume that P is an arbitrary non-empty set of objects. Let

$$\sqsubset := \{\langle x, y\rangle \in P \times P \mid x \text{ is a part of } y\}\,.$$

Then, expressing the intuitions formerly stated, we have

$$\forall_{x,y \in P}(x \sqsubset y \Rightarrow y \not\sqsubset x)\,. \tag{as-$\sqsubset$}$$

In consequence

$$\forall_{x \in P}\, x \not\sqsubset x\,. \tag{irr-$\sqsubset$}$$

Outline

Formalization

Transitive fragments of part of relation

Suppose that $Z \subseteq P$ and let $\sqsubset|_Z$ be a restriction of $\sqsubset$ to Z

$$\sqsubset|_Z := \sqsubset \cap\, Z \times Z\,.$$

There is at least one non-empty subset Z of P for which $\sqsubset|_Z$ is transitive

$$\forall_{x,y,z \in Z}(x \sqsubset y \wedge y \sqsubset z \Rightarrow x \sqsubset z)\,.$$

It is enough to take one- or two-element subset of P.

Formalization
Maximal elements

We introduce the family of subsets of P in which *part of* relation is transitive.

$$\mathcal{T}_P := \{Z \in \mathcal{P}(P) \mid \sqsubset|_Z \text{ is transitive}\}\,. \tag{def-$\mathcal{T}_P$}$$

$\mathcal{T}_P$ is of finite character. Thus by Tukey's lemma it contains at least one element which maximal with respect to set-theoretical inclusion.
Let

$$\mathcal{M}_P := \{Z \in \mathcal{T}_P \mid \text{for no } Y \in \mathcal{T}_P,\ Z \subsetneq Y\}\,. \tag{1}$$

We may say that elements of $\mathcal{M}_P$ are sets of objects with a mereological meaning of *part of* relation.

Outline

Classical mereology
First axioms

Let M be a non-empty set, $\sqsubset \subseteq M \times M$ and let $\langle M, \sqsubset\rangle$ satisfy the following conditions

$$\forall_{x,y\in M}(x \sqsubset y \Rightarrow y \not\sqsubset x)\,, \tag{L1}$$

$$\forall_{x,y,z\in M}(x \sqsubset y \wedge y \sqsubset z \Rightarrow z \sqsubset z)\,. \tag{L2}$$

We have from (L1) that

$$\forall_{x\in M} x \not\sqsubset x\,.$$

Outline

Classical mereology
Auxiliary relations

We introduce three auxiliary binary relations in M.

$$x \sqsubseteq y \overset{\text{df}}{\iff} x \sqsubset y \lor x = y\,, \qquad \text{(def-}\sqsubseteq\text{)}$$

x is an ingrediens of y,

$$x \circ y \overset{\text{df}}{\iff} \exists_{z \in M}(z \sqsubseteq x \land z \sqsubseteq y)\,, \qquad \text{(def-}\circ\text{)}$$

x overlaps (is compatible with) y,

$$x \perp y \overset{\text{df}}{\iff} \neg x \circ y\,, \qquad \text{(def-}\perp\text{)}$$

x is exterior to (is incompatible with) y.

Outline

Classical mereology

Definition of a mereological class - intuitions

There are two basic definitions - of mereological sum and mereological fusion. Suppose that we have some set Z of objects.

- an object x is a mereological sum of all elements of Z iff every element of Z is an ingrediens of x and every ingrediens of x overlaps some element z from Z,
- an object x is a fusion of all elements of Z iff for every object y, y overlaps x iff y overlaps some element z from Z.

Classical mereology

Formal definition of a mereological sum and fusion

We define binary relations: **Sum** $\subseteq M \times \mathcal{P}(M)$ and **Fus** $\subseteq M \times \mathcal{P}(M)$. Let $Z \subseteq M$.

$$x \ \mathbf{Sum}\ Z \overset{\mathrm{df}}{\iff} \forall_{z \in Z} z \sqsubseteq x \wedge \forall_{y \in M}(y \sqsubseteq x \Rightarrow \exists_{z \in M} y \circ z)\,, \qquad \text{(def-}\mathbf{Sum}\text{)}$$

x is a mereological sum of Z

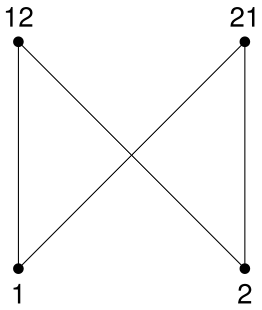
$$x \ \mathbf{Fus}\ Z \overset{\mathrm{df}}{\iff} \forall_{y \in M}(y \circ x \iff \exists_{z \in Z} y \circ z)\,, \qquad \text{(def-}\mathbf{Fus}\text{)}$$

$$(\iff \forall_{y \in M}(y \perp x \iff \forall_{z \in Z} y \perp z))\,,$$

x is a fusion of Z

Classical mereology
Mereological sum and fusion – comparison

It follows from (L1), (L2) and definition introduced so far that **Sum** $\subseteq$ **Fus**, however not vice versa, that is **Fus** $\not\subseteq$ **Sum**.



$$12\ \mathbf{Fus}\ \{1, 2, 12, 21\}$$

$$\neg 12\ \mathbf{Sum}\ \{1, 2, 12, 21\}$$

Classical mereology

Mereological sum and fusion – some properties

Fact

$\neg\exists_{x\in M} x$ **Sum** $\emptyset$ *and* $\neg\exists_{x\in M} x$ **Fus** $\emptyset$ *(roughly speaking — there is no such thing as an empty mereological set).*

$$x \textbf{ Sum } Z \overset{\text{df}}{\iff} \forall_{z\in Z} z \sqsubseteq x \wedge \forall_{y\in M}(y \sqsubseteq x \Rightarrow \exists_{z\in M} y \circ z)\,,$$

$$x \textbf{ Fus } Z \overset{\text{df}}{\iff} \forall_{y\in M}(y \circ x \iff \exists_{z\in Z} y \circ z)\,.$$

Classical mereology

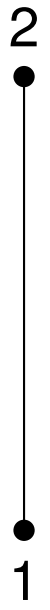
Mereological sum and fusion – some properties

Moreover, (L1) and (L2) are not enough to prove that for a given set of object there exists exactly one mereological sum (fusion) of its objects.

$2 \textbf{ Sum } \{1\}$

$1 \textbf{ Sum } \{1\}$

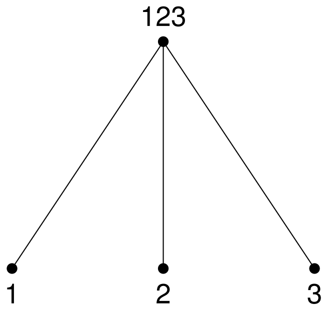
$2 \neq 1$



Classical mereology

Mereological sum vs. supremum

$$\mathbf{Sum} \subseteq \sup_{\sqsubseteq} \quad \text{but} \quad \sup_{\sqsubseteq} \nsubseteq \mathbf{Sum}$$



For example:

$$123 \sup_{\sqsubseteq} \{1, 2\}$$
$$\neg 123 \ \mathbf{Sum} \ \{1, 2\} .$$

Classical mereology

Mereological sum – some basic consequences

$$x \,\mathbf{Sum}\, \{x\}\,, \tag{2}$$

$$x \,\mathbf{Sum}\, \{y \in M \mid y \sqsubseteq x\}\,, \tag{3}$$

$$\{y \in M \mid y \sqsubset x\} \neq \emptyset \Rightarrow x \,\mathbf{Sum}\, \{y \in M \mid y \sqsubset x\}\,. \tag{4}$$

Outline

Classical mereology
Mereological sum – axiom of uniqueness

After Leśniewski we introduce the following (second-order) axiom.

$$\forall_{x,y\in M}\forall_{Z\in\mathcal{P}_+(M)}(x \textbf{ Sum } Z \wedge y \textbf{ Sum } Z \Rightarrow x = y)\,. \tag{L3}$$

In consequence, by the fact that y **Sum** $\{y\}$,

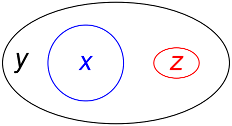
$$x \textbf{ Sum } \{y\} \Rightarrow x = y\,. \tag{S}$$

Classical mereology

Mereological sum – some consequences of (L3)

Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ be a structure with a binary irreflexive relation $\sqsubset$ and with $\sqsubseteq$, $\perp$ and **Sum** defined as above. If $\mathfrak{M}$ satisfies (L3), then

$$\forall_{x,y \in M}(x \sqsubset y \Rightarrow \exists_{z \in M}(z \sqsubset y \wedge z \perp x)). \qquad \text{(WSP)}$$



Proof.

(L3) entails (S). Suppose towards contradiction, that $x \sqsubset y$ but $\forall_{z \in M}(z \sqsubset y \Rightarrow z \circ x)$. Therefore y **Sum** $\{x\}$, which means $y = x$ (by (S)) and contradicts irreflexivity of $\sqsubset$. □

Classical mereology

The smallest element (usually) does not exist

Let $\mathfrak{M} = \langle M, \sqsubset \rangle$ satisfy (WSP). Then

$$\mathrm{Card}\, M > 1 \iff \exists_{x,y \in M} x \perp y\,.$$

Proof.

Suppose that (a) $\mathrm{Card}\, M > 1$. Therefore there are x_1 and x_2 such that (b) $x_1 \neq x_2$. Either $x_1 \perp x_2$ or $x_1 \circ x_2$. Let $z \in M$ be such that $z \sqsubseteq x_1$ and $z \sqsubseteq x_2$. By (b), either $z \sqsubset x_1$ or $z \sqsubset x_2$. Thus, by (WSP), either there is z_0 such that $z_0 \sqsubset x_1$ and $z_0 \perp z$ or $z_0 \sqsubset x_2$ and $z_0 \perp z$. □

In consequence

$$\exists_{x \in M} \forall_{y \in M} x \sqsubseteq y \iff \mathrm{Card}\, M = 1\,. \qquad (\nexists 0)$$

Classical mereology

Axiom of sum existence. Mereological structures

Natural question: for which subsets of M should we require existence of its mereological sum?

$$\forall_{Z \in \mathcal{P}_+(M)} \exists_{x \in M} x \ \textbf{Sum}\ Z\,. \tag{L4}$$

Immediate consequence

$$\exists_{x \in M} \forall_{y \in M} y \sqsubseteq x\,. \tag{$\exists$1}$$

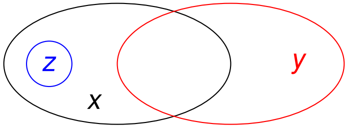
Any structure $\mathfrak{M} = \langle M, \sqsubset \rangle$ satisfying (L1)–(L4) is called a mereological structure.

Outline

Classical mereology
Basic properties of mereological structures

In every mereological structure

$$\forall_{x,y\in M}(x \sqsubseteq y \Rightarrow \exists_{z\in M}(z \sqsubseteq x \wedge z \perp y))\,. \tag{SSP}$$



$$\mathbf{Sum} = \mathbf{Fus} \tag{5}$$

$$x\ \mathbf{Sum}\ Z \iff Z \neq \emptyset \wedge x \operatorname{sup}_{\sqsubseteq} Z\,. \tag{6}$$

Classical mereology

A couple of last remarks

- Boolean complete algebras vs. mereological structures
- applications of mereology: point-free topology, point-free geometry, logic (definition of some special consequence relations), category theory, set theories with urelements.

Thank you for your attention!