The independence of the covering numbers of the splitting tree forcing ideal and the Sacks forcing ideal

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Countably splitting analytic sets

Definition

A set $A \subseteq 2^\omega$ is called countably splitting *iff* for each countable $B \subseteq [\omega]^\omega$ there is a $x \in A$ such that x splits every $b \in B$
i.e.: $|x^{-1}[0] \cap b| = \aleph_0$ and $|x^{-1}[1] \cap b| = \aleph_0$

Theorem (O.Spinas 2004)

For $A \subseteq 2^\omega$ analytic holds: A countably splitting iff there exists a splitting tree p with $[p] \subseteq A$

Splitting Tree Forcing

Definition

A perfect Tree $p \subseteq 2^{<\omega}$ is called splitting *iff*
$\forall \sigma \in p \exists K(\sigma) \forall n \geq K(\sigma) \exists \tau_0, \tau_1 \supseteq \sigma : |\tau_0|, |\tau_1| \geq n \wedge \tau_0(n) = 0 \wedge \tau_1(n) = 1$

We let S denote the Forcing consisting of all splitting trees ordered by inclusion. The generic real added by this Forcing splits all $b \in [\omega]^\omega$ of the ground model.

Definition

Let $I(S) := \{X \subseteq 2^\omega \mid \forall p \in S \exists q \leq p : [q] \cap X = \emptyset\}$ be the Ideal generated by the splitting tree forcing.

Main Theorem

Theorem (Wy 2011)

$V^S_{\omega_2} \models Cov(I(\mathbb{S})) < Cov(I(S))$

$V^S_{\omega_2} \models \text{Cov}(I(S)) = \omega_2$ is shown analogously to $V^{\mathbb{S}}_{\omega_2} \models \text{Cov}(I(\mathbb{S})) = \omega_2$ which has been done by Judah, Miller and Shelah, so the focus of this talk will be to show that $V^S_{\omega_2} \models \text{Cov}(I(\mathbb{S})) = \omega_1$

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2. if $\emptyset \in H$ then $F(\dot{\emptyset})$ is a front of $p(\emptyset)$
3. for all $\beta \in H$ with $\emptyset < \beta$ we have that $p \restriction \beta \Vdash F(\dot{\beta})$ is a Front of $p(\beta)$

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Let $\alpha \in OR$; $p, q \in S_\alpha$; $\dot{F}, \dot{G}$ regarding H-Fronts
Define $(q, \dot{G}) \leq_H (p, \dot{F}) :\Leftrightarrow$

1. if $\emptyset \in H$ then $(q(\emptyset), \dot{G}(\emptyset)) \leq (p(\emptyset), \dot{F}(\emptyset))$
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Definition

Let $\alpha \in OR$; $p \in S_\alpha$; $H \in [supp(p)]^{<\omega}$ and $\dot{F}$ be a H-Front for p.
We say that p is $(H, \dot{F})$-decided iff: For all $\overline{\sigma} \in^H (2^{<\omega})$:either

1. $\forall \beta \in H : (p \restriction \beta) \restriction (\overline{\sigma} \restriction \beta) \Vdash \overline{\sigma}(\beta) \in \dot{F}(\beta)$ or
2. $\exists \gamma \in H \forall \beta \in H; \beta < \gamma : (p \restriction \beta) \restriction (\overline{\sigma} \restriction \beta) \Vdash \overline{\sigma}(\beta) \in \dot{F}(\beta)$
$\wedge\ (p \restriction \gamma) \restriction (\overline{\sigma} \restriction \gamma) \Vdash \overline{\sigma}(\gamma) \notin \dot{F}(\gamma)$

Injective continuous reading

Lemma (Wy 2011)

Let $\alpha \in OR$; $p \in S_\alpha$ *and* $\dot{x}$ *a* S_α*-Name for a real such that for all* $\xi < \alpha$:
$p \Vdash \dot{x} \notin V_\xi$
Then there exits a $q \leq p$ *and a Sequence* $< H_i; \dot{F}_i; k_i | i \in \omega >$ *such that*

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2. $\dot{F}_i$ *is a* H_i*-Front for* q*;*
3. $(q, \dot{F}_{i+1}) \leq_{H_{i+1}} (q, \dot{F}_i)$*;*
4. q *is* $(H_i, \dot{F}_i)$*-decided;*
5. $k_i \in \omega$*;* $k_{i+1} > k_i$ *for all* $i \in \omega$

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Injective continuous reading

Lemma (continued)

...and there exists a Family $\{\xi_{\overline{\sigma}} \in 2^{<\omega} | \overline{\sigma} \in \bigcup_{i\in\omega} \dot{F}_i\}$ *such that*

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1. *for every* $i \in \omega$ *and* $\overline{\sigma} \in \dot{F}_i$*:* $q_{\overline{\sigma}} \Vdash \xi_{\overline{\sigma}} \subseteq \dot{x}$

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2. *for every* $i \in \omega$ *and* $\overline{\sigma} \in \dot{F}_i$: $length(\xi_{\overline{\sigma}}) \geq k_i$

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1. *for every* $i \in \omega$ *and* $\overline{\sigma} \in \dot{F}_i$: $q_{\overline{\sigma}} \Vdash \xi_{\overline{\sigma}} \subseteq \dot{x}$
2. *for every* $i \in \omega$ *and* $\overline{\sigma} \in \dot{F}_i$: $length(\xi_{\overline{\sigma}}) \geq k_i$
3. *for two indices* $\overline{\sigma}, \overline{\sigma}'$ *that are incompatible in at least one coordinate we have* $\xi_{\overline{\sigma}} \perp \xi_{\overline{\sigma}'}$

Proof of the Theorem

Reminder:
We now want to show that $V^S_{\omega_2} \models \mathrm{Cov}(I(\mathbb{S})) = \omega_1$

Proof.

- Let $< q_\lambda \mid \lambda < \omega_2 >$ an enumeration of some arbitrary dense Set $D \in V^S_{\omega_2}$ of Sacks conditions
- Try to build a matrix $< q_{\xi\lambda} | \xi < \omega_1; \lambda < \omega_2 >$ with $q_{\xi\lambda} \leq q_\lambda$ such that for any new real $x \in V^S_{\omega_2}$ there is a row ξ with $x \notin [q_{\xi\lambda}]$ for all $\lambda < \omega_2$
- The sets $X_\xi := 2^\omega \setminus \bigcup_{\lambda<\omega_2} [p_{\xi\lambda}]$ are Sacks Ideal sets that cover all (new) reals

Proof of the Theorem

Proof continued.

- assign to each new real x a condition of the generic filter that witnesses the injective continuous reading of names and the regarding family $P := \{\xi_{\overline{\sigma}} \mid \overline{\sigma} \in \bigcup_{i\in\omega} \dot{F}_i\}$ as in the previous lemma
- "throw away" the information about the exact position of the coordinates by collapsing the support of that condition to some $\delta < \omega_1$ and adjusting the $\dot{F}_i$ and the the $\overline{\sigma}$ accordingly
- this will give us one of ω_1-many functions $f : \overline{\sigma} \mapsto \xi_{\overline{\sigma}}$ with $[f^{-1}x]$ being a sequence of generic reals of the support (without the information where exactly they occur)

Proof of the Theorem

Proof continued.

Thin out the the q_λ from the dense set to some $q_{f\lambda}$ in the following way:

- Case 1: $q_\lambda \nsubseteq P$. You can easily find a perfect $q_{f\lambda} \leq q_\lambda$ with $[q_{f\lambda}] \cap [P] = \emptyset$. It follows that every new real that has the function f assigned to it is not an element of $[q_{f\lambda}]$. So we are done

Proof of the Theorem

Proof continued.

- Case 2: $q_\lambda \subseteq P$. Let $q_\lambda \in V_\gamma^S$ By a fusion argument you can thin out q_λ to $q'_{f\lambda}$ such that for each coordinate $\xi < \delta$ we have $[\pi_\xi f^{-1} q'_{f\lambda}] \subseteq V_\gamma$. This means that every real that has assigned the function f to it and is an element of $[q'_{f\lambda}]$ is introduced in an intermediate model V_α with $\alpha \leq \gamma$
  So if you pick a $q_{f\lambda} \leq q'_{f\lambda}$ such that its closure is disjoint to V_γ, the closure wont contain any reals with the function f assigned to it

□

...more results

Theorem (Baumgartner Laver 1979)

Every real in $V^{\mathbb{S}}_{\omega_2}$ is refined by a ground model real

Corollary

$V^{\mathbb{S}}_{\omega_2} \models Cov(I(S)) = \omega_1$

Corollary

$V^{\mathbb{S}}_{\omega_2} \models Cov(I(S)) < Cov(I(\mathbb{S}))$

open questions

- $\mathrm{add}(I(S)) < \mathrm{add}(I(\mathbb{S}))$?
- $\mathrm{add}(I(S)) > \mathrm{add}(I(\mathbb{S}))$?
- Is there a (non-natural) amoeba forcing for the splitting tree forcing that is proper and minimal?
- ZFC $\vdash \mathrm{Cov}(I(S)) \leq \mathrm{Cov}(M)$?

Thank You for Your attention