Two sets

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Let \((X, +)\) be any uncountable Polish abelian group and let \(I \subseteq \mathcal{P}(X)\) s.t

- \(I\) is \(\sigma\)-ideal with a Borel base and
- \(I\) contains all singletons and
- \(I\) translation invariant.

The \(\sigma\)-ideal \(I\) is nice if has properties as above.

Let \(\mathcal{B}_+(I) = \text{Borel}(X) \setminus I\) be set of all \(I\)-positive Borel sets.

\(\text{Perf}(X)\) stands for set of all perfect subsets of \(X\).

In most part of presentation \(X\) is a real plane \(\mathbb{R}^2\) and \(+\) denotes adding vectors.
Definition (Cardinal coefficients)

Let $X$ - Polish space and $I \subseteq \mathcal{P}(X)$ be $\sigma$ ideal as above. Then for any $\mathcal{F} \subset I$ let

$$cov(\mathcal{F}, I) = \min\{|A| : A \subset \mathcal{F} \land \bigcup A = X\}$$

$$cov_h(\mathcal{F}, I) = \min\{|A| : A \subset \mathcal{F} \land (\exists B \in \mathcal{B}_+(I)) \bigcup A = B\}$$

$Lines$ be the set of all lines in $\mathbb{R}^2$.
$L$ $\sigma$-ideal of null sets and
$K$ $\sigma$-ideal of all meager subsets of $X$.

Fact
$\text{cov}_h(Lines, L) = 2^\omega$, $\text{cov}_h(Lines, K) = 2^\omega$. 
**Definition (Cardinal coefficients)**

Let $X$ - Polish space and $I \subseteq P(X)$ be $\sigma$ ideal as above. Then for any $\mathcal{F} \subset I$ let

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$Lines$ be the set of all lines in $\mathbb{R}^2$.

$L_\sigma$ $\sigma$-ideal of null sets and

$K_\sigma$ $\sigma$-ideal of all meager subsets of $X$.

**Fact**

$\text{cov}_h(Lines, L) = 2^\omega$, $\text{cov}_h(Lines, K) = 2^\omega$. 
Definition (Two-set)
A subset $X \subseteq \mathbb{R}^2$ of the real plane is a two-set iff meets every line in exactly two points.

Theorem (Mazurkiewicz 1914)

There exist a two-set.
Two-sets with a Hamel base

Definition
Let $X$ be any uncountable Polish space. We say that a set $A \subseteq X$ is completely $I$-nonmeasurable iff

$$(\forall B \in \mathcal{B}_+(X))\ A \cap B \neq \emptyset \land B \cap A^c \neq \emptyset$$

Note that if $I = [X]^{\leq \omega}$ then $A$ is Bernstein set. Moreover if $I = \mathbb{L}$ then $A$ is completely nonmeasurable subset of $X$.

Theorem
Let $I \subseteq P(\mathbb{R}^2)$ be any nice $\sigma$-ideal with $\text{cov}_h(\text{Lines}, I) = 2^\omega$. Then there exists a two point set $A \subseteq \mathbb{R}^2$, that is completely $I$-nonmeasurable Hamel base.

Corollary
There exists a two point set $A \subseteq \mathbb{R}^2$, that is completely nonmeasurable Hamel base.
Two-sets with a Hamel base

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Corollary
There exists a two point set \( A \subseteq \mathbb{R}^2 \), that is completely nonmeasurable Hamel base.
Proof

Let \( \{ L_\xi : \xi < c \} \) all straight lines in the plane \( \mathbb{R}^2 \),
let \( \{ B_\xi : \xi < c \} \) be an enumeration of all positive Borel sets in \( \mathbb{R}^2 \)
\( \{ h_\xi : \xi < c \} \) be a Hamel base of \( \mathbb{R}^2 \) over \( \mathbb{Q} \).
Define \( \{ A_\xi : \xi < c \} \) of subsets of \( \mathbb{R}^2 \) such that for every \( \xi < c \),

1. \( |A_\xi| < \omega \);
2. \( \bigcup_{\zeta \leq \xi} A_\zeta \) does not have three collinear points;
3. \( \bigcup_{\zeta \leq \xi} A_\zeta \) contains precisely two points of \( L_\xi \);
4. \( B_\xi \cap \bigcup_{\zeta \leq \xi} A_\zeta \neq \emptyset \);
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Then, the set \( A = \bigcup_{\xi < c} A_\xi \) will have desired property.
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Marczewski ideal

Definition
If $X$ is Polish space then $A \subseteq \mathbb{R}$ is $s_0$-Marczewski iff

$$(\forall P \in Perf(X))(\exists Q \in Perf(X)) \ Q \subseteq P \land Q \cap A = \emptyset$$

and $A \subseteq \mathbb{R}$ is $s$-Marczewski (s-measurable) iff

$$(\forall P \in Perf(X))(\exists Q \in Perf(X)) \ Q \subseteq P \land (Q \cap A = \emptyset \lor Q \subseteq A).$$

Theorem
There exists a two point set $A \subseteq \mathbb{R}^2$, that is $s_0$-Marczewski.
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**Theorem**
There exists a two point set $A \subseteq \mathbb{R}^2$, that is $s_0$-Marczewski.
Definition (Partial two-set)

We say that $A \subseteq \mathbb{R}^2$ is a partial two-set iff meets every line at most two times.

It is well known that the unit circle is a partial two-set which cannot be extended to two-set.
Theorem
There exists a two point set $A \subseteq \mathbb{R}^2$, that is s-nonmeasurable. Moreover $A$ contains a subset of the unit circle of full outer measure.

Proof.
Let $C$ be a unit circle, $\text{Lines} = \{l_\xi : \xi < c\}$ and $\mathcal{B}_+(C, \mathbb{L}) = \{P_\xi : \xi < c\}$. Define a sequences $\{A_\xi : \xi < c\}$ $\{y_\xi : \xi < c\}$ s.t. for every $\xi < c$
1. $|A_\xi| < \omega$;
2. $\bigcup_{\zeta \leq \xi} A_\zeta$ does not contain three collinear points;
3. $\bigcup_{\zeta \leq \xi} A_\zeta$ contains precisely two points of $L_\xi$;
4. $P_\xi \cap \bigcup_{\zeta \leq \xi} A_\zeta \neq \emptyset$;
5. $y_\xi \in P_\xi$;
6. $A_\xi \cap \{y_\zeta : \zeta \leq \xi\} = \emptyset$.

Then $A = \bigcup_{\xi < c} A_\xi$ is required set.
Iso-covering set

**Definition (\(\kappa\)-set)**
We say that \(A \subseteq \mathbb{R}^2\) is a \(\kappa\)-set iff every line meets exactly in \(\kappa\)-points.

**Definition (\(\kappa\)-iso cov)**
We say that \(A \subseteq \mathbb{R}^2\) is \(\kappa\)-iso cov set iff for every \(X \in [\mathbb{R}^2]^\kappa\) there exist isometry \(g\) on the real plane such that \(g[X] \subseteq A\).
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Decomposition of two-sets

**Theorem**

*Every two-set can be decomposed onto two bijections of the real line $\mathbb{R}$.*

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*There exists a null and meager two-set $A \subseteq \mathbb{R}^2$ s.t. every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ cannot be contained in $A$.*

and

**Theorem**

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Two-set vs. Luzin set

Fact
Any two-set cannot be
- Bernstein set
- Luzin set and
- Sierpiński set.

Proof.
1) Each line $L$ is a perfect set such that $|A \cap L| = 2$, so $A$ cannot be Bernstein.
2) Let $M$ be a perfect meager subset of $\mathbb{R}$. Then $M \times \mathbb{R}$ is meager and $|(M \times \mathbb{R}) \cap A| = 2|M| = c$.
3) Let $N$ be a perfect null subset of $\mathbb{R}$. Then $N \times \mathbb{R}$ is null and $|(N \times \mathbb{R}) \cap A| = 2|N| = c$. □
Theorem

Assume CH then

1. there exists partial two point set $A$ that is Luzin set,
2. there exists partial two point set $B$ that is Sierpiński set.
Definition (ad family)
The set $\mathcal{A} \subset [\omega]^\omega$ is almost disjoint family (ad) iff any two distinct members of $\mathcal{A}$ has finite intersection.
$\mathcal{A}$ is (mad) iff $\mathcal{A}$ is a maximal respect to the $\subseteq$.

Definition (Eventually different functions)
We say that $\mathcal{A} \subseteq \omega^\omega$ is eventually different family in Baire space $\omega^\omega$ iff every two distinct members $x, y \in \mathcal{A}$ are equal only on the finite subset of the $\omega$.
$\mathcal{A}$ is maximal eventually different family iff $\mathcal{A}$ is a maximal respect to the inclusion relation.
Theorem (CH)

Let $h : \mathbb{R} \to \omega^\omega$ be a standard Borel bijection. Then there exist the partial two-point set $A \subseteq \mathbb{R}^2$ on the real plane such that

$$\{ h(\pi_i(x)) \in \omega^\omega : x \in A \land i \in \{0, 1\} \} - \text{max. eventually different.}$$

where $\pi_i$ are projections onto $i$-th axis.

Remark

The same result is about mad family instead maximal eventually different functions family.
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Remark

The same result is about mad family instead maximal eventually different functions family.
Proof

Consider sequence \((M_\alpha : \alpha < \omega_1)\) the increasing continuous chain of the countable subset of \(\mathbb{R}\) with \(\mathbb{R} \subseteq \bigcup_{\alpha < \omega_1} M_\alpha\). Let us construct the transfinite sequence \((A_\alpha, F_\alpha) : \alpha < \omega_1\) s.t.

1. \((\forall \alpha < \omega_1)\) \(A_\alpha = \{x_\xi \in \mathbb{R}^2 : \xi < \alpha\} \in M_\alpha\) is a partially two-point set,

2. \((\forall \alpha < \omega_1)\) \(F_\alpha = \{h \circ \pi_i(x_\xi) : x_\xi \in A_\alpha \land i \in \{1, 2\}\}\) forms family eventually different functions,

3. \((\forall \alpha < \omega_1)\) \((\forall u \in M_\alpha \cap (\omega^\omega \setminus F_\alpha))\) \((\exists v \in F_{\alpha+1})\) \(|u \cap v| = \omega\).
Proof

Correctness: let us assume that $A_\alpha$ is build at $\alpha < \omega_1$ step. Enumerate $(\omega^\omega \setminus F_\alpha) \cap M_\alpha = \{y_n : n \in \omega\}$. In $H_\kappa$ model we can to construct the sequence $x_n : n \in \omega$ as follows if $\{x_k : k < n\}$ is build then we can choose $x_n$ such that

for any $u$ if $u \in F_\alpha \cup \{x_k : k < n\}$ then

$$|h(x_n) \cap u| < \omega \land (h^{-1}(x_n), h^{-1}(y_n)) \notin W_\alpha$$

where $W_\alpha = \{l \in \text{lines} : |l \cap Z_\alpha| = 2\}$ and

$$Z_\alpha = \{(x, y) : x, y \in h^{-1}[F_\alpha] \cup \{x_k : k < n\} \cup \{y_n : n \in \omega\} \land x \neq y\}.$$ 

Using properties (1), (2) and (3) it is easy to show that $A = \bigcup_{\alpha < \omega_1} A_\alpha$ fulfil the assertion of this Theorem.
Here we adopt the proof of the Kunen Theorem about existence of the indestructible mad family (see [Ku] for example).

**Theorem**

*It is consistent with ZFC theory that \( \neg CH \) and there exists partial two-set for which the image of the set of all coordinates forms the mad family size \( \omega_1 \) by standard bijection \( h : \mathbb{R} \to P(\omega) \).*
Theorem

It is consistent that $\neg CH$ and

$$ (\exists C \in [\mathbb{R}^2]^{\omega_2})(\exists A \in \mathbb{L})(\exists D_1 \in [C]^{\omega_1}) $$

s.t

$$ A + D_1 = \mathbb{R}^2 \land C \text{ is partial two-set.} $$

Moreover the set $C$ is a Luzin set.
Proof

Let $V$ - ground model with $CH$.
Now $\mathbb{P} = Fn(\omega_2, 2)$ be forcing adding indenpendetly $c_\alpha : \alpha < \omega_2$ Cohen points on the $\mathbb{R}^2$.
If $\alpha < \beta < \gamma < \omega_2$ then $c_\gamma$ is Cohen over $c_\alpha$ and $c_\beta$.
Then $c_\gamma \notin I_{\alpha, \beta}$ where $c_\alpha, c_\beta \in I_{\alpha, \beta}$ forms line $I_{\alpha, \beta} \in K$.
We see that $C = \{c_\alpha : \alpha < \omega_2\}$ is partial two set.
C is Luzin:

Let $G$ be $\mathbb{P}$-generic ultrafilter over $V$. Take $x \in \omega^\omega \cap V[G]$ be any Borel code for a meager subset of $\mathbb{R}^2$. Find $I \in [\omega_2]^{\omega}$ and nice name $\tilde{x} \in V^{\text{Fn}(I,2)}$ for $x$. Define

$$G_I = \{ p \in \text{Fn}(I,2) : p \in G \} \quad G_{\omega_2 \setminus I} = \{ p \in \text{Fn}(\omega \setminus I,2) : p \in G \}.$$

Then

- $V[G] = V[G_I][G_{\omega_2 \setminus I}]$
- $x \in V[G_I]$ and
- for any $\alpha \in \omega_2 \setminus I$ $c_\alpha \in V[G] \setminus V[G_I]$ is Cohen over $V[G_I]$.

Then $C \cap \#x \subseteq \{ c_\alpha : \alpha \in I \}$ is countable.
Proof

Consider a Marczewski decomposition $A \cup B = \mathbb{R}^2$ where $A \in \mathcal{L}$, $B \in \mathcal{K}$ and $A \cap B = \emptyset$.

Choose $D \in V[G] \cap [\omega_2]^{\omega_1}$ and $x \in V[G]$.

Then by c.c.c. of $Fn(\omega_2, 2)$ we have

1. $\exists D_1 \in V \cap [\omega_2]^{\omega_1}$ $D \subseteq D_1$ and
2. $(\exists I \in [\omega_2]^\omega)$ $V[G] = V[G_I][G_{\omega_2 \setminus I}]$ and $x \in V[G_I]$
3. $(\forall \alpha \in D \setminus I)$ $c_\alpha \in A - \{x\}$

Then finally in $V[G]$ we have $\mathbb{R}^2 \subseteq A - C_{D_1 \setminus I}$ where $C_{D_1 \setminus I} = \{c_\alpha \in C : \alpha \in D_1 \setminus I\}$. 
Thank You
References


Mazurkiewicz S., O pewnej mnogości płaskiej, która ma z każdą prostą dwa i tylko dwa punkty wspólne (Polish), Comptes Rendus des Séances de la Société des Sciences et Lettres de Varsovie 7 (1914), 382–384; French transl.: Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs, in: Stefan Mazurkiewicz, Traveaux de Topologie et ses Applications (K. Borsuk et al., eds.), PWN, Warsaw, 1969, pp. 46–47.
