A simple Boolean algebra with complicated space of measures

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- Let such an independent family $\mathcal{J}$ be faithfully indexed as $\{N_b : b \in \mathfrak{B}\}$.

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In other words, $\mathfrak{A}$ is freely generated by G_b modulo $G_{b_1} \wedge \ldots \wedge G_{b_k} = 0$ whenever $b_1 \wedge \ldots \wedge b_k = 0$.

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μ_n's distinguish elements of $\mathfrak{A}$ and moreover $\widehat{\mu_n}$'s distinguish continuous functions on K: if $g, h \in C(K)$ and

$$\int_K h \, \mathrm{d}\widehat{\mu_n} = \int_K g \, \mathrm{d}\widehat{\mu_n},$$

for every n then $g = h$.

Theorem (Mägerl-Namioka)

Given any algebra $\mathfrak{C}$, the space $P(\mathfrak{C})$ is separable iff there is there is a sequence $\nu_n \in P(\mathfrak{A})$ such that for every $a \in \mathfrak{A}^+$, $\nu_n(a) \geq 1/2$ for some n.

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Theorem (APR,Talagrand under CH)

There is a compact space K such that $C(K)^*$ is weak*-separable while the unit ball in $C(K)^*$ is not weak*-separable.

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The following implications hold

$$\boxed{(B_{X^*}, weak^*) \text{ sep.}} \Rightarrow \boxed{B_X \in Ba(X)} \Rightarrow \boxed{(X^*, weak^*) \text{ sep.}}$$

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- We do not know if this implies $B_{C(K)} \in Ba(C(K))$...