A simple Boolean algebra with complicated space of measures

Grzegorz Plebanek (Uniwersytet Wrocławski)

joint work with A. Avilés and J. Rodríguez
(Universidad de Murcia)

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Let $B$ be the measure algebra of the product measure $\lambda$ on $\mathbb{N}$. For every $b \in B$, $b = B \cdot B_0$, where $B_0 \in \mathcal{B}$ or $(2^I)$, $I \subseteq \mathbb{N}$ countable. In particular, $|B| = \aleph_0$. We denote still by $\lambda$ the measure on $B$. $B$ has density $\aleph_0$ in the Frechet-Nikodym distance $(a, b) \rightarrow \lambda(a \Delta b)$. There is an independent family $J = \{N_\xi : \xi < \mathbb{N}\}$ of subsets of $\mathbb{N}$; this means that for any finite and disjoint $s$, $t \subseteq \mathbb{N}$, $\bigcap_{\xi \in s} N_\xi \cap \bigcap_{\xi \in t} (\mathbb{N} \setminus N_\xi) \neq \emptyset$. Let such an independent family $J$ be faithfully indexed as \{\{N_b : b \in B\}\}.
Let $\mathcal{B}$ be the measure algebra of the product measure $\lambda$ on $2^\mathbb{N}$.
Algebra $\mathcal{B}$ and independent family in $\mathbb{N}$

- Let $\mathcal{B}$ be the measure algebra of the product measure $\lambda$ on $2^c$.
- For every $b \in \mathcal{B}$, $b = B^\cdot$, where $B = B_0 \times 2^c \setminus I$, $B_0 \in \text{Bor}(2^I)$, $I \subseteq c$ countable.
Let $\mathcal{B}$ be the measure algebra of the product measure $\lambda$ on $2^\mathbb{c}$.

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In particular, $|\mathcal{B}| = \mathbb{c}$. 
Algebra $\mathcal{B}$ and independent family in $\mathbb{N}$

- Let $\mathcal{B}$ be the measure algebra of the product measure $\lambda$ on $2^c$.
- For every $b \in \mathcal{B}$, $b = B^*$, where $B = B_0 \times 2^{c \setminus I}$, $B_0 \in \text{Bor}(2^I)$, $I \subseteq c$ countable.
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$\mathcal{B}$ has density $c$ in the Frechet-Nikodym distance $(a, b) \rightarrow \lambda(a \Delta b)$. 

**Algebra $\mathcal{B}$ and independent family in $\mathbb{N}$**
Algebra $\mathcal{B}$ and independent family in $\mathbb{N}$

- Let $\mathcal{B}$ be the measure algebra of the product measure $\lambda$ on $2^\omega$.
- For every $b \in \mathcal{B}$, $b = B^\cdot$, where $B = B_0 \times 2^{c \setminus l}$, $B_0 \in \text{Bor}(2^l)$, $l \subseteq c$ countable.
- In particular, $|\mathcal{B}| = \omega$.
- We denote still by $\lambda$ the measure on $\mathcal{B}$.
- $\mathcal{B}$ has density $\omega$ in the Frechet-Nikodym distance $(a, b) \mapsto \lambda(a \triangle b)$.
- There is an independent family $\mathcal{J} = \{N_\xi : \xi < \omega\}$ of subsets of $\mathbb{N}$; this means that for any finite and disjoint $s, t \subseteq \omega$,

$$\bigcap_{\xi \in s} N_\xi \cap \bigcap_{\xi \in t} (\mathbb{N} \setminus N_\xi) \neq \emptyset.$$
Let $\mathcal{B}$ be the measure algebra of the product measure $\lambda$ on $2^c$.

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Let such an independent family $\mathcal{J}$ be faithfully indexed as $\{N_b : b \in \mathcal{B}\}$. 
Algebra $\mathcal{A}$ (recall $\mathcal{J} = \{N_b : b \in \mathcal{B}\}$)

Work in the simple product $\mathcal{B} \times \mathcal{N}$; if $a \in \mathcal{B} \times \mathcal{N}$ then $a = (a(n))$ if $n \in \mathcal{N}$.

Define $G_b \in \mathcal{B} \times \mathcal{N}$ as $G_b(n) := b$ if $n \in \mathcal{N}_b$, 0 otherwise.

Definition $A$ is the subalgebra in $\mathcal{B} \times \mathcal{N}$ generated by all $G_b$, $b \in \mathcal{B}$.

In other words, $A$ is freely generated by $G_b$ modulo $G_{b_1} \wedge ... \wedge G_{b_k} = 0$ whenever $b_1 \wedge ... \wedge b_k = 0$. 

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Work in the simple product $\mathcal{B}^\mathbb{N}$; if $a \in \mathcal{B}^\mathbb{N}$ then $a = (a(n))_{n \in \mathbb{N}}$. 

Define $G_b \in \mathcal{B}^\mathbb{N}$ as $G_b(n) :=
\begin{cases}
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Algebra $\mathcal{A}$, $K = ULT(\mathcal{A})$ and the Banach space $C(K)$
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Let $P(\mathcal{A})$ be the space of all finitely additive measures on $\mathcal{A}$. $P(\mathcal{A})$ is a compact space as a subspace of $[0, 1]^\mathcal{A}$. 

Every $\mu \in P(\mathcal{A})$ defines uniquely a regular probability measure $\hat{\mu}$ on $K$. We have $\mu_n \in P(\mathcal{A})$ defined as $\mu_n(a) = a(n)$ for $a \in \mathcal{A}$. $\mu_n$'s distinguish elements of $\mathcal{A}$ and moreover $\hat{\mu}_n$'s distinguish continuous functions on $K$: if $g, h \in C(K)$ and $\int_K h \, d\hat{\mu}_n = \int_K g \, d\hat{\mu}_n$, for every $n$ then $g = h$. 

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Theorem (Mägerl-Namioka)

Given any algebra $\mathcal{C}$, the space $P(\mathcal{C})$ is separable iff there is a sequence $\nu_n \in P(\mathcal{A})$ such that for every $a \in \mathcal{A}^+$, $\nu_n(a) \geq 1/2$ for some $n$. 

Lemma

The space $P(\mathcal{A})$ is not separable.

Proof. $P(\mathcal{B})$ is not separable. $\mathcal{B}$ can be identified with $\mathcal{B}_1 \subseteq \mathcal{B}_\infty$ consisting of constant sequences. For every $a \in \mathcal{B}_1^+$ there is $a' \in \mathcal{A}^+$ such that $a' \leq a$. This and theorem above imply that $P(\mathcal{A})$ is not separable.

Theorem (APR, Talagrand under CH)

There is a compact space $K$ such that $C(K)^*$ is weak$^*$-separable while the unit ball in $C(K)^*$ is not weak$^*$-separable.
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Theorem (APR, Talagrand under CH)

There is a compact space $K$ such that $C(K)^*$ is weak*-separable while the unit ball in $C(K)^*$ is not weak*-separable.
Let $X$ be a Banach space and $X^*$ its dual. Let $\mathcal{Ba}(X)$ denote the least $\sigma$-algebra making all $x^*$ measurable. $\mathcal{Ba}(X)$ is generated by all half-spaces $\{x \in X : x^*(x) \leq r\}$, $r \in \mathbb{R}$, $x^* \in X^*$. Note that the norm $\|\cdot\| : X \to \mathbb{R}$ is $\mathcal{Ba}(X)$-measurable iff $B_X = \{x \in X : \|x\| \leq 1\} \in \mathcal{Ba}(X)$, iff $B_X$ can be made of countably many halfspaces. Recall that the weak$^*$-topology on $X^*$ is the topology of pointwise convergence on $X$, i.e. a typical neighbourhood of $0 \in X^*$ is of the form $\{x^* \in X^* : |x^*(x_1)| < \varepsilon, \ldots, |x^*(x_k)| < \varepsilon\}$. The following implications hold: $(B_X^*, \text{weak}^*)$ sep. $\Rightarrow$ $B_X \in \mathcal{Ba}(X) \Rightarrow$ $(X^*, \text{weak}^*)$ sep.
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Note that the norm $\| \cdot \| : X \to \mathbb{R}$ is $Ba(X)$-measurable iff

$$B_X = \{x \in X : \|x\| \leq 1\} \in Ba(X),$$

iff $B_X$ can be made of countably many halfspaces.
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The following implications hold

$$(B_{X^*}, \text{weak}^*) \text{ sep.} \Rightarrow B_X \in Ba(X) \Rightarrow (X^*, \text{weak}^*) \text{ sep.}$$
The problem

Let $K = \text{ULT}(A)$; is $B \in C(K)$ in $\text{Ba}(C(K))$?

Some partial results

$B \in C(K)$ is not in the $\sigma$-algebra generated by $\hat{\mu}_n$'s.

Given $n$, there is $\mu_2^n \in P(A)$ such that $\mu_2^n(G_b) = (\lambda(b))^2$,

whenever $n \in \mathbb{N}^b$.

Given a simple function $g \in C(K)$, the condition $\|g\| \leq 1$

can be expressed in terms of $\hat{\mu}_n$ and $\hat{\mu}_2^n$ using countable

quantifiers.

We do not know if this implies $B \in C(K) \in \text{Ba}(C(K))$.

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- $B_{C(K)}$ is not in the $\sigma$-algebra generated by $\mu_n$'s.
- Given $n$, there is $\mu_n^2 \in P(\mathcal{A})$ such that
  \[ \mu_n^2(G_b) = (\lambda(b))^2, \]
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