Covering properties of ideals

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joint work with

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Theorem (M. Elekes)

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $(A_n)_{n\in\omega} \in \mathcal{A}^\omega$ be a μ-a.e. infinite-fold cover of X, that is,

$$\left\{x \in X : \{n \in \omega : x \in A_n\} \text{ is finite}\right\} \text{ has } (\mu\text{-})\text{measure } 0.$$

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Then there exists a set $S \subseteq \omega$ such that $\lim_{n\to\infty} \frac{|S\cap n|}{n} = 0$ and $(A_n)_{n\in S}$ is also a μ-a.e. infinite-fold cover of X.

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Question (Elekes)

Possible generalizations? Applications?

The $\mathcal{J}$-covering property

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if for every I-a.e. infinite-fold cover $(A_n)_{n\in\omega} \in \mathcal{A}^\omega$, there is an $S \in \mathcal{J}$ such that $(A_n)_{n\in S}$ is also an I-a.e. infinite-fold cover.

Remarks on the definition

$(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property:

IF $(A_n)_{n\in\omega} \in \mathcal{A}^\omega$ is an (I-a.e.) infinite-fold cover,

THEN $\exists\, S \in \mathcal{J}$ $(A_n)_{n\in S}$ is also an I-a.e. infinite-fold cover.

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Remarks

(1) Elekes' theorem in this context: If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, then $(\mathcal{A}, \mathrm{Null}(\mu))$ has the $\mathcal{Z}$-covering property where $\mathcal{Z} = \{S \subseteq \omega : \lim_{n\to\infty} \frac{|S\cap n|}{n} = 0\}$ is the *density zero ideal* (a tall $F_{\sigma\delta}$ P-ideal).

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(2) If $\mathcal{J}$ is not tall (i.e. there is an $H \in [\omega]^\omega$ s.t. $\mathcal{J} \upharpoonright H = [H]^{<\omega}$), then there is no $(\mathcal{A}, I)$ with the $\mathcal{J}$-covering property.

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Remarks

(3) If $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property, then $(\mathcal{A}[I], I)$ also has this property where $\mathcal{A}[I]$ is the "I-completion of $\mathcal{A}$", that is

$$\mathcal{A}[I] = \{B \subseteq X : \exists\ A \in \mathcal{A}\ A \triangle B \in I\}.$$

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(4) If $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property, then for all $Y \in \mathcal{A} \setminus I$ the pair $(\mathcal{A} \restriction Y, I \restriction Y)$ also has this property.

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Remarks

(5) If $I_1 \subseteq I_2$ are ideals on X and $(\mathcal{A}, I_1)$ has the $\mathcal{J}$-covering property, then $(\mathcal{A}, I_2)$ also has the $\mathcal{J}$-covering property.

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(5) If $I_1 \subseteq I_2$ are ideals on X and $(\mathcal{A}, I_1)$ has the $\mathcal{J}$-covering property, then $(\mathcal{A}, I_2)$ also has the $\mathcal{J}$-covering property.

(6) If $(\mathcal{A}, I)$ has the $\mathcal{J}_0$-covering property and $\mathcal{J}_0 \leq_{\mathrm{KB}} \mathcal{J}_1$, i.e.

$$\exists\, f : \omega \xrightarrow{\text{fin-to-one}} \omega\ \forall\, S \in \mathcal{J}_0\ f^{-1}[S] \in \mathcal{J}_1,$$

then $(\mathcal{A}, I)$ has the $\mathcal{J}_1$-covering property as well.

Analytic uniformity

$(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property:

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Reformulation: $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property iff

for every $(\mathcal{A}, \mathrm{Borel}([\omega]^\omega))$-measurable $F\colon X \to [\omega]^\omega$, there is an $S \in \mathcal{J}$ such that $\{x \in X : |F(x) \cap S| = \omega\} \in I^*$.

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The case $I = \{\emptyset\}$ $\sim$ star-uniformity of $\mathcal{J}$

$(\mathcal{P}(X), \{\emptyset\})$ has the $\mathcal{J}$-covering property iff $|X| < \mathrm{non}^*(\mathcal{J})$ where $\mathrm{non}^*(\mathcal{J}) =$

$$\min\{|\mathcal{H}| : \mathcal{H} \subseteq [\omega]^\omega \text{ and } \nexists\, A \in \mathcal{J}\ \forall\, H \in \mathcal{H}\ |A \cap H| = \omega\}.$$

Notation

If $\mathcal{A}$ is clear from the context (usually it will be the Borel σ-algebra on a Polish space), then we will simply write:

I has the $\mathcal{J}$-covering property.

Covering property vs. forcing indestructibility

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Assume that $\mathcal{J}$ is a tall ideal on ω and $\mathbb{P}$ is a forcing notion. We say that $\mathcal{J}$ is ***$\mathbb{P}$-indestructible*** if the ideal in $V^{\mathbb{P}}$ generated by $\mathcal{J}$ is tall,

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Theorem

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Proof: Assume on the contrary that there is a $\mathbb{P}_I$-name $\dot{Y}$ s.t. $\Vdash_{\mathbb{P}_I} \dot{Y} \in [\omega]^{\omega}$ and $B \Vdash_{\mathbb{P}_I} \forall\, A \in \mathcal{J}\ |\dot{Y} \cap A| < \omega$ for some $B \in \mathbb{P}_I$.

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Proof (continued): For each $n \in \omega$ let

$$Y_n = f^{-1}\left[\{S \in [\omega]^\omega : n \in S\}\right] \in \mathrm{Borel}(X).$$

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- $I = \mathrm{NWD}$ is the ideal of nowhere dense subsets of ω^ω;
- $I = \mathcal{K}_\sigma$ is the σ-ideal (σ-)generated by compact sets, in other words, $\mathcal{K}_\sigma = \langle \{g \in \omega^\omega : g \leq^* f\} : f \in \omega^\omega \rangle_{\mathrm{id}}$ where $g \leq^* f$ iff $\forall^\infty n\; g(n) \leq f(n)$.

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Then I does not have the $\mathcal{J}$-covering property for any $\mathcal{J}$.

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Then I does not have the $\mathcal{J}$-covering property for any $\mathcal{J}$.

Proof: Consider the following infinite-fold cover of ω^ω:

$$A_n = \{f \in \omega^\omega : f(n) \neq 0\} \cup \{g \in \omega^\omega : \forall^\infty n\, g(n) = 0\}.$$

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If $S, \omega \setminus S \in [\omega]^\omega$, then $\omega^\omega \setminus \limsup_{n \in S} A_n = \liminf_{n \in S}(\omega^\omega \setminus A_n) = \{f \in \omega^\omega : \forall^\infty n \in S\, f(n) = 0\}$ is dense and not in $\mathcal{K}_\sigma$.

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The summable ideal

$$\mathcal{I}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n+1} < \infty \right\}$$

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For every $k \in \omega$, the set $\bigcup_{i\geq k} A_{(n_i,m_i)}$ is dense and open.
Consequently, $\limsup_{(n,m)\in S} A_{(n,m)} = \bigcap_{k\in\omega} \bigcup_{i\geq k} A_{(n_i,m_i)}$ is a dense G_δ set, hence it is residual (i.e. co-meager).

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Answer for Question (1): No

The ideal $\mathrm{Fin} \otimes \mathrm{Fin} = \{A \subseteq \omega \times \omega : \forall^\infty n \in \omega \; |(A)_n| < \omega\}$ (a tall $F_{\sigma\delta\sigma}$ non P-ideal) and $\mathcal{ED}$

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Answer: No

Consider $X = \mathbb{R}$ and let

$$I = \{A \subseteq \mathbb{R} : A \cap (-\infty, 0] \text{ is meager and } A \cap [0, \infty) \text{ is null}\}.$$

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Then I does not have the $\mathcal{I}_{1/n}$-covering property but for each infinite-fold Borel cover $(A_n)_{n\in\omega}$ of X, there is an $S \in \mathcal{I}_{1/n}$ such that $\limsup_{n\in S} A_n \in \mathcal{M}((-\infty, 0])^* \subseteq I^+ (= \mathcal{P}(\mathbb{R}) \setminus I)$.

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Assume furthermore that $\mathcal{J}$ is a P-ideal and I does not have the $\mathcal{J}$-covering property.

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$$\forall\, X \in \mathrm{Borel}(G) \setminus I \;\; D + X \in I^*.$$

Assume furthermore that $\mathcal{J}$ is a P-ideal and I does not have the $\mathcal{J}$-covering property. Then there is an infinite-fold Borel cover $(A_n)_{n\in\omega}$ of G such that $\limsup_{n\in S} A_n \in I$ for each $S \in \mathcal{J}$.

Assume I is translation invariant...

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Remark

$\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy the conditions of the theorem with any countable dense subsets of $\mathbb{R}$ (resp. $\mathbb{R}^2$).

Thank you!