Local connectedness, cardinal invariants, and images of H^*

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January 30, 2012

Outline

$\mathbb{H}^*$, the Stone-Cech remainder of $[0,\infty)$

Notation:

- continuum = compact connected Hausdorff space.
- $\mathbb{H} = [0,\infty)$, $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$.

Prologue

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Local connectedness

- What we are interested in is the following: Let X be a continuous image of $\mathbb{H}^*$. Then there is a compactification Y of $\mathbb{H}$ with remainder X. Suppose that X satisfies a certain cardinal characteristic p. Then, when is it possible to embed Y in a locally connected continuum which satisfies p? That is, we want to study the (possible) preservation of cardinal invariants on X in local connectifications of Y.
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Cardinal invariants

Recall the definition of the Suslin number (or cellularity) of a space X.

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To motivate our results, we start from folklore, sketching the proof of the following fact.

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Sketch of proof. X is a compact metric space, hence $\exists$ continuous map $f : C \to X$ from the Cantor set onto X. Assume $\{0, 1\} \subset C \subset [0, 1]$. Let $\mathcal{G}$ be the decomposition of $[0, 1]$ into fibers of f, $\{f^{-1}(x) : x \in X\}$, and singletons.
The quotient space, $Y = [0, 1]/\mathcal{G}$ is a metrizable locally connected continuum containing a homeomorphic copy of X.
$Y \setminus X$ is a union of countably many pairwise disjoint open arc, and obviously $\overline{c}(Y) = \overline{c}(X)$.

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Theorem

Suppose no κ^+-Souslin tree exists. Then for every continuum X which is an image of $\mathbb{H}^*$ and which has $\overline{c}(X) = \kappa$, we can embed any compactification of $\mathbb{H}$ with X as remainder in a locally connected continuum Y with $\overline{c}(Y) = \kappa$.

Corollary

Under the Souslin hypothesis, for each Suslinian continuum X we can embed any compactification of $\mathbb{H}$ with X as remainder in a locally connected Suslinian continuum.

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Theorem

Under the negation of the Souslin hypothesis, there is a Suslinian continuum X (namely a compact, connected Suslin line), such that no compactification of $\mathbb{H}$ with X as remainder can be embedded in a locally connected Suslinian continuum.

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1. Motivation
2. Preliminary results
3. Techniques
4. One way of looking at things

Non-metric compacta as inverse limits

Theorem [Mardesic]

A non-metric compactum X is homeomorphic to the inverse limit of a well-ordered inverse system $(X_\alpha, f_\alpha^\beta, \kappa)$, where each factor space X_α is compact with $w(X_\alpha) < w(X)$, each bonding map f_α^β is surjective, and $\kappa \leq w(X)$. If, moreover, X is locally connected, we may choose the inverse system to be such that each bonding mapping is also monotone.

Of course, if X is locally connected, then each factor space X_α is locally connected.

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Scepin spectral theorem

We will make use of the spectral theorem of Scepin in our analysis of non-metric continua.

Theorem [Scepin]

Let $\{X_\alpha, p^\beta_\alpha, \kappa\}$ and $\{Y_\alpha, q^\beta_\alpha, \kappa\}$ be two continuous well-ordered inverse systems, where

1. κ is an uncountable regular cardinal, and
2. $w(X_\alpha) < \kappa$ for every $\alpha < \kappa$,

and denote by X and Y the respective inverse limits.
Then for any map $f : X \to Y$, there exists a clubset $C \subseteq \kappa$ and maps $f_\alpha : X_\alpha \to Y_\alpha$, $\alpha \in C$, such that $f = \varprojlim\{f_\alpha, \alpha \in C\}$. If f were a homeomorphism, then each f_α would also be a homeomorphism.

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Let T be a Souslin tree. We may assume that T satisfies the following additional properties:

1. $\forall t \in T, succ(t)$ is uncountable,
2. the level T_0 is infinite, and $\forall t \in T, immsucc(t)$ is infinite,
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