Universal homogeneous structures in ZFC

(Joint work with Antonio Avilés)

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Classical Fraïssé theory

The setup: Fraïssé class

- $\mathscr{F}$ is a class of finitely generated structures.
- Joint Embedding Property: Given $X, Y \in \mathscr{F}$, there is $Z \in \mathscr{F}$ such that both $X \hookrightarrow Z$ and $Y \hookrightarrow Z$.
- Amalgamation Property: Given embeddings $i \colon Z \hookrightarrow X$, $j \colon Z \hookrightarrow Y$ with $Z, X, Y \in \mathscr{F}$, there exists $W \in \mathscr{F}$ such that for some embeddings the diagram

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$$\begin{array}{ccc} Z & \xrightarrow{\ i\ } & X \\ {\scriptstyle j}\downarrow & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

commutes.

Fraïssé theorem

$$\sigma\mathscr{F} := \Big\{ \bigcup_{n\in\omega} X_n \colon \{X_n\}_{n\in\omega} \subseteq \mathscr{F} \text{ is a chain} \Big\}$$

Theorem

Let $\mathscr{F}$ be a countable Fraïssé class. Then there exists a unique, up to isomorphism, countable structure $U = \operatorname{Flim}\mathscr{F}$, satisfying the following conditions.

1. $U \in \sigma\mathscr{F}$.
2. *Given $\mathscr{F}$-structures $X \subseteq Y$, given an embedding $e \colon X \hookrightarrow U$, there exists an embedding $f \colon Y \hookrightarrow U$ such that $f \restriction X = e$.*
3. *Every $\mathscr{F}$-structure embeds into U.*

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Homogeneity & Universality

Theorem (Homogeneity)

Let $\mathscr{F}$ be a countable Fraïssé class and let $U = \operatorname{Flim}\mathscr{F}$. Then for every substructures $E, F \subseteq U$ such that $E, F \in \mathscr{F}$, every isomorphism $h \colon E \to F$ extends to an automorphism $H \colon U \to U$.

Theorem (Universality)

For every $X \in \sigma\mathscr{F}$ there exists an embedding $X \hookrightarrow U$.

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What to do if $\mathscr{F}$ is uncountable?

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Finite metric spaces.

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The setup:

A cardinal $\kappa \geqslant \aleph_0$ is given.

- $\mathscr{F}$ has both the Joint Embedding and the Amalgamation Property.
- Each member of $\mathscr{F}$ should have size $< \kappa$.
- $\mathscr{F}$ is closed under unions of chains of length $< \kappa$.

Quite often, $|\mathscr{F}| = \kappa^{<\kappa}$.

Typical assumption:

$$\kappa^{<\kappa} = \kappa$$

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Let $\mathscr{F}$ be a κ-Fraïssé class, where κ is a regular cardinal and $|\mathscr{F}| = \kappa$. Then there exists a unique, up to isomorphism, structure $U_\kappa = \operatorname{Flim} \mathscr{F}$ of size κ, satisfying the following conditions.

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Fact

There are $\mathfrak{c}$ many finite metric spaces.

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If $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ then

$$\mathscr{M} := \{\langle X, d\rangle : \ \langle X, d\rangle \text{ is a metric space and } |X| < \mathfrak{c}\}$$

is a $\mathfrak{c}$-Fraïssé-Jónsson class.

It may happen that

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But we'd like to have a homogeneous structure of size $\mathfrak{c}$ ...

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Theorem (Avilés & Brech 2011)

Assume $\mathrm{cf}(\mathfrak{c}) = \mathfrak{c}$. *There exists a unique Boolean algebra* $\mathfrak{B}$ *satisfying the following conditions.*

1. $|\mathfrak{B}| = \mathfrak{c}$ *and* $\mathfrak{B}$ *is tightly* σ*-filtered.*
2. *Given Boolean algebras* $A \leqslant C$ *such that* $|C| < \mathfrak{c}$ *and* A *is a* σ*-subalgebra of* C*, every embedding* $e \colon A \to \mathfrak{B}$ *extends to an embedding* $f \colon C \to \mathfrak{B}$.

Fact

$\mathfrak{c} = \aleph_1 \implies \mathfrak{B} \approx \mathscr{P}(\omega)/_{[\omega]^{<\omega}}$.

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In the $\aleph_2$*-Cohen model,* $\mathfrak{B} \approx \mathscr{P}(\omega)/_{[\omega]^{<\omega}}$.

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Assume $\mathrm{cf}(\mathfrak{c}) = \mathfrak{c}$. *There exists a unique Banach space* $\mathfrak{X}$ *satisfying the following conditions.*

1. $\mathrm{dens}(\mathfrak{X}) = \mathfrak{c}$ *and* $\mathfrak{X}$ *is tightly* σ*-filtered.*
2. *Given Banch spaces* $A \leqslant C$ *such that* $\mathrm{dens}(C) < \mathfrak{c}$ *and* C *is a tight extension of* A*, every embedding* $e\colon A \to \mathfrak{X}$ *extends to an embedding* $f\colon C \to \mathfrak{X}$.

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Main ingredient: pushouts

A pushout square

$$
\begin{array}{ccc}
y & \xrightarrow[g']{} & w \\
{\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\
z & \xrightarrow[f]{} & x
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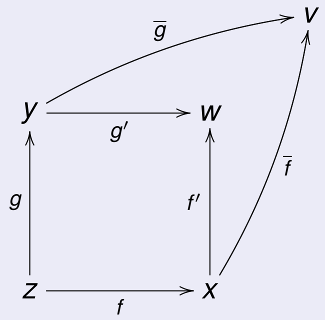

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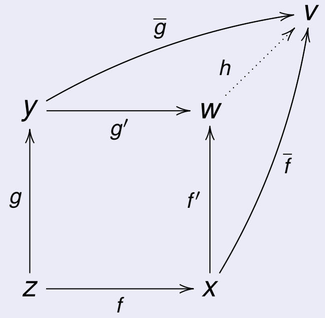

Claim

Let $A \leqslant B$ be Boolean algebras such that B is finitely generated over A. The following properties are equivalent:

1. *A is a σ-subalgebra of B.*
2. *There exist countable Boolean algebras $C \leqslant D$ and embeddings for which*

$$\begin{array}{ccc} A & \xrightarrow{\subseteq} & B \\ \uparrow & & \uparrow \\ C & \xrightarrow[\subseteq]{} & D \end{array}$$

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Our goal

Assumptions:

- $\mathscr{F}$ is a class of "small" objects.
- $\mathscr{F}$ has pushouts.

Goal:

We'd like to have for each $\kappa \geqslant |\mathscr{F}|$ an object $\mathfrak{X}_\kappa$ satisfying:

1. $\mathfrak{X}_\kappa$ has size κ and is the union (limit) of a directed system of objects from $\mathscr{F}$.
2. $\mathscr{F} \subseteq \mathscr{L}$, where $\mathscr{L}$ is a class of arbitrarily "large" objects of the same language.
3. $\mathfrak{X}_\kappa$ contains isomorphic copy of every object from $\mathscr{F}$.
4. $\mathfrak{X}_\kappa$ is $\mathscr{F}$-homogeneous.

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Definition

An embedding $e\colon A \to B$ is **$\mathscr{F}$-tight** if there are $a, b \in \mathscr{F}$ and embeddings $a \to b$, $a \to A$, $b \to B$, such that

$$\begin{array}{ccc} A & \xrightarrow{\ e\ } & B \\ \uparrow & & \uparrow \\ a & \longrightarrow & b \end{array}$$

is a pushout square.

Definition

Let $A, B \in \mathscr{L}$. We say that B is $\mathscr{F}$-tightly filtered over A if there exists a continuous chain $\{A_\xi\}_{\xi \leqslant \delta} \subseteq \mathscr{L}$ such that

1. $A_0 = A$, $A_\delta = B$,
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Definition

$$\mathscr{T}_\kappa := \{X \in \mathscr{L} : X \text{ is } \mathscr{F}\text{-tightly filtered of size } \leqslant \kappa\}.$$

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2. $A_\xi \subseteq A_{\xi+1}$ is $\mathscr{F}$-tight for each $\xi < \delta$.

Definition

$$\mathscr{T}_\kappa := \{X \in \mathscr{L} \colon X \text{ is } \mathscr{F}\text{-tightly filtered of size } \leqslant \kappa\}.$$

$$\mathscr{T}_{<\kappa} := \bigcup_{\lambda<\kappa} \mathscr{T}_\lambda.$$

Terminology from algebra

$\mathscr{F}$-tight filtration = relative $\mathscr{F}$-cell complex

Reference:

M. HOVEY, *Model Categories*, Mathematical Surveys and Monographs 63, AMS, 1999

Example

Let $\mathscr{F}$ be the class of all countable Boolean algebras.
Then $\mathscr{T}_\kappa$ is the class of tightly σ-filtered Boolean algebras of size $\leqslant \kappa$.

Example

Let $\mathscr{F}$ be the class of all finite Boolean algebras.
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Main result

Theorem

Let $\mathscr{F}$ be as before, and let $\kappa \geqslant |\mathscr{F}| + \aleph_0$.
There exists a unique object $\mathfrak{X}_\kappa \in \mathscr{T}_\kappa$ satisfying the following conditions:

1. *For every object in $X \in \mathscr{T}_\kappa$ there exists a relative $\mathscr{F}$-cell complex from X to $\mathfrak{X}_\kappa$.*
2. *Given an $\mathscr{F}$-tight inclusion $A \subseteq B$ with $A, B \in \mathscr{T}_{<\kappa}$, every embedding $e\colon A \to \mathfrak{X}_\kappa$ extends to an embedding $f\colon B \to \mathfrak{X}_\kappa$.*

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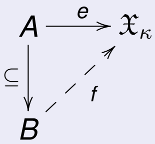

Theorem

Assume $A, B \subseteq \mathfrak{X}_\kappa$ are such that $A, B \in \mathscr{T}_{<\kappa}$ and $\mathfrak{X}_\kappa$ is $\mathscr{F}$-tightly filtered over both A and B.

Then every isomorphism between A and B extends to an automorphism of $\mathfrak{X}_\kappa$.

Corollary

The object $\mathfrak{X}_\kappa$ is $\mathscr{F}$-homogeneous.

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Assume $A, B \subseteq \mathfrak{X}_\kappa$ *are such that* $A, B \in \mathscr{T}_{<\kappa}$ *and* $\mathfrak{X}_\kappa$ *is* $\mathscr{F}$*-tightly filtered over both A and B.*
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The object $\mathfrak{X}_\kappa$ *is* $\mathscr{F}$*-homogeneous.*

About the proof

Theorem

Assume κ is regular. Then the category $\mathscr{T}_{<\kappa}$ (with relative $\mathscr{F}$-complexes) has a κ-Fraïssé sequence.

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The singular case

Definition

A PL-functor is a covariant functor $F\colon L \to \mathscr{F}$ such that L is a tight lattice and for every $a, b \in L$ the following diagram is a pushout in $\mathscr{L}$.

$$\begin{array}{ccc} F(b) & \longrightarrow & F(a \vee b) \\ \uparrow & & \uparrow \\ F(a \wedge b) & \longrightarrow & F(a) \end{array}$$

Theorem

Given $X \in \mathscr{L}$, the following properties are equivalent:

1. *$X \in \mathscr{T}_\kappa$ for some κ,*
2. *$X = \lim F$, where $F\colon L \to \mathscr{F}$ is a PL-functor.*

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Definition

A PL-functor $F\colon L \to \mathscr{F}$ is κ-Fraïssé if:

- given $a \in L$, given $S \in [L]^{<\kappa}$ such that $a < s$ for $s \in S$, given an embedding $f\colon F(a) \to y$ with $y \in \mathscr{F}$, there exists $b > a$ such that $b \wedge s = a$ for every $s \in S$, and the bonding map

$$F_a^b\colon F(a) \to F(b)$$

is isomorphic to f.

Theorem

A κ-Fraïssé PL-functor exists as long as $\kappa \geqslant |\mathscr{F}|$. It is unique, up to an isomorphism.

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Fact

$\mathfrak{X}_\kappa = \lim F$, *where F is a* κ*-Fraïssé PL-functor.*

Application: a short proof of Shchepin's theorem

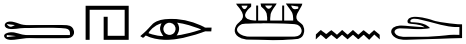
Theorem (Shchepin 1976)

A projective Boolean algebra if free if and only if it is homogeneous with respect to density.

Short proof.

1. Fix a projective algebra $\mathfrak{B}$, homogeneous by density.
2. Check that given a tight embedding $A \subseteq A[x]$ with $|A| < |\mathfrak{B}|$, given an embedding $e\colon A \to \mathfrak{B}$, there exists an embedding $f\colon A[x] \to \mathfrak{B}$ such that $f \restriction A = e$.

□



THE END

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