# Applications of the Löwenheim-Skolem theorem. Part III

## Quidquid latine dictum sit, altum videtur

K. P. Hart

Faculty EEMCS
TU Delft

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# Outline

# Chainability

**Definition**

A continuum, $X$, is chainable if every (finite) open cover $\mathcal{U}$ has an open chain-refinement $\mathcal{V}$, i.e., $\mathcal{V}$ can be written as $\{V_i : i < n\}$ such that $V_i \cap V_j \neq \emptyset$ iff $|i - j| \leqslant 1$.

$[0, 1]$ is chainable; the circle $S^1$ is not.

# Span zero

**Definition**

A continuum, $X$, has xxx span zero if every subcontinuum $Z$ of $X \times X$ that satisfies yyy intersects the diagonal $\{\langle x, x\rangle : x \in X\}$.

| xxx | yyy | symbol |
|---|---|---|
| ... | $\pi_1[Z] = \pi_2[Z]$ | $\sigma X$ |
| semi | $\pi_1[Z] \subseteq \pi_2[Z]$ | $\frac{1}{2}\sigma X$ |
| surjective | $\pi_1[Z] = \pi_2[Z] = X$ | $s\sigma X$ |
| surjective semi | $\pi_2[Z] = X$ | $s\frac{1}{2}\sigma X$ |

$[0, 1]$ has all spans zero, $S^1$ has all spans non-zero

# An implication

**Theorem**

*In a chainable continuum all spans are zero.*

**Proof.**

If $Z$ is a continuum that is disjoint from the diagonal then take a chain cover $\{V_i : i < n\}$ such that $Z \cap \bigcup_{i<n} V_i^2 = \emptyset$.
Then $Z \subseteq \bigcup_{i<j} V_i \times V_j$ or $Z \subseteq \bigcup_{i>j} V_i \times V_j$.
In either case $Z$ does not satisfy any of the mapping properties. □

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# The problem

**Question (Lelek)**

What about the converse?

This was an important problem in metric continuum theory.

But it makes non-metric sense as well.

# Implications

$$\begin{array}{ccc} \sigma X = 0 & \leftarrow & \frac{1}{2}\sigma X = 0 \\ \downarrow & & \downarrow \\ s\sigma X = 0 & \leftarrow & s\frac{1}{2}\sigma X = 0 \end{array}$$

or, contrapositively

$$\begin{array}{ccc} \sigma X > 0 & \rightarrow & \frac{1}{2}\sigma X > 0 \\ \uparrow & & \uparrow \\ s\sigma X > 0 & \rightarrow & s\frac{1}{2}\sigma X > 0 \end{array}$$

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# $\mathbb{H}^*$ is not chainable

$\mathbb{H} = [0, \infty)$ and $\mathbb{H}^*$ is its Čech-Stone remainder.
For $i = 0, 1, 2, 3$ put

$$U_i = \bigcup_{n=0}^{\infty} \left(4n + i - \frac{5}{8}, 4n + i + \frac{5}{8}\right)$$

and

$$O_i = \operatorname{Ex} U_i \cap \mathbb{H}^*$$

where $\operatorname{Ex} U = \beta\mathbb{H} \setminus \operatorname{cl}(\mathbb{H} \setminus U)$
(the largest open set in $\beta\mathbb{H}$ that intersects $\mathbb{H}$ in $U$).

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# $\mathbb{H}^*$ is not chainable

The open cover $\{O_0, O_1, O_2, O_3\}$ of $\mathbb{H}^*$ does not have a chain refinement — nice exercise, but a bit convoluted.

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# The spans of $\mathbb{H}^*$

It would be nice if some of the spans of $\mathbb{H}^*$ were zero:
we'd have a non-metric counterexample to Lelek's conjecture.

However: consider $f : \mathbb{H} \to \mathbb{H}$, defined by $f(x) = x + 1$,
and its extension $\beta f : \beta\mathbb{H} \to \beta\mathbb{H}$,
and that extension's restriction $f^* : \mathbb{H}^* \to \mathbb{H}^*$.

Its graph witnesses that the surjective span of $\mathbb{H}^*$ is non-zero and hence so are the other three.

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# Other candidates

Consider $\mathbb{M} = \omega \times [0, 1]$ and its Čech-Stone compactification $\beta\mathbb{M}$.

The extension $\beta\pi : \beta\mathbb{M} \to \beta\omega$ of the projection $\pi : \mathbb{M} \to \omega$ divides $\beta\mathbb{M}$ into continua.

For $u \in \omega^*$ we punt $\mathbb{I}_u = \beta\pi^{\leftarrow}(u)$.

What can we say about the spans of the $\mathbb{I}_u$?

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# The span of $\mathbb{I}_u$

**Theorem**

*The span of $\mathbb{I}_u$ is non-zero.*

The proof is like that for $\mathbb{H}^*$: the continua $\mathbb{I}_u$ contain subcontinua that are quite similar to $\mathbb{H}^*$ and they allow an analogue of the graph of $x \mapsto x + 1$.

# The other spans of $\mathbb{I}_u$

**Theorem (CH)**

*The surjective span of $\mathbb{I}_u$ is non-zero.*

The proof is more involved and can best be illustrated with a picture.

Here the speaker draws an instructive picture on the blackboard.

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# Why is this interesting?

$\mathbb{I}_u$ has a (very) nice base for its closed sets: the *ultrapower* of $2^{\mathbb{I}}$ by the ultrafilter $u$.

Remember:

The ultrapower of a lattice $L$ is formed as follows.

First take the power $L^{\mathbb{N}}$, with pointwise operations.

Then say $f \sim_u g$ if $\{n : f(n) = g(n)\} \in u$.

The quotient structure $\prod_u L = L^{\mathbb{N}}/{\sim_u}$ is the *ultrapower* of $L$ by the ultrafilter $u$.

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# Why is this interesting?

The big theorem on ultrapowers:
$L$ and $\prod_u L$ are elementary equivalent.

Even:
The 'obvious' embedding of $L$ into $\prod_u L$ is an elementary embedding.

# Chainability is not first-order

Chainability is, just like covering dimension, a property of every/some lattice base for the closed sets.
(Shrink-and-swell again.)

Now then, . . . , $2^{\mathbb{I}}$ satisfies 'chainability' but $\prod_u 2^I$ does not, so

unlike the dimensions, chainability is not expressible in first-order terms *in the language of lattices*.

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# A formula for chainability

The natural formulation is an $L_{\omega_1,\omega}$-formula.

$$(\forall u_1)(\forall u_2)(\forall u_3)(\forall u_4) \Big((u_1 \cup u_2 \cup u_3 \cup u_4 = X) \to \bigvee_{n\in\omega} \Phi_n(u_1, u_2, u_3, u_4)\Big)$$

where $\Phi_n(u_1, u_2, u_3, u_4)$ expresses that $\{u_1, u_2, u_3, u_4\}$ has an $n$-element chain refinement.

It (indeed) suffices to consider four-element open covers only.

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# Span zero is . . .

The status of span zero is not clear: it is either

- not a property reducible to bases or
- not first-order.

This would make a nice research problem.

# Reflection

**Theorem**

*Any counterexample to Lelek's problem can be converted into a metrizable counterexample.*

**Proof.**

Let $X$ be a counterexample, let $L \prec 2^X$ (an elementary sublattice). Then $wL$ is a metrizable counterexample. □

Not quite . . . because of what we have just seen.

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# Solution: Use Set Theory

Let $\theta$ be 'suitably large' and let $M \prec H(\theta)$ be a countable elementary substructure and let $L = M \cap 2^X$.

**Theorem**

*In this situation:*

- $wL$ *is chainable iff* $X$ *is chainable*
- $wL$ *has span zero iff* $X$ *has span zero (any kind)*

# Proof for Chainability

Chainability is now first-order; we can quantify over the finite subsets of $2^X$ and finite ordinals.

Furthermore, one needs only consider covers and refinements that belong to a certain base.

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# Span zero

Key observation: let $K = M \cap 2^{X \times X}$, then $wK = wL \times wL$.

This gives the easy part: if there is a ‘bad’ continuum in $X \times X$ then there is one in $M$ and it is equally bad in $wL \times wL$.

For the converse ...

# Span zero, continued

... if $Z \subseteq wL \times wL$ is 'bad' then there is an equally bad continuum in $X \times X$ that maps onto $Z$.

Easier said than constructed: the difficulty lies in the fact that $K$ is not (necessarily) an elementary substructure of $2^{wK}$.

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# Span zero, the real argument

Apply Shelah's Ultrapower theorem: take a cardinal $\kappa$, an ultrafilter $u$ on $\kappa$ and an isomorphism $h : \prod_u (2^{X \times X}) \to \prod_u wK$ (which can be taken to be the identity on $K$).

How does that help?

For that we need some topology.

# Dualizing ultrapowers

Take a compact Hausdorff space $Y$ with a lattice base $B$. Also take a cardinal $\kappa$ and an ultrafilter $u$ on $\kappa$.

Consider $\beta(\kappa \times Y)$. We have two maps

- $p_\kappa : \beta(\kappa \times Y) \to \beta\kappa$ (the extension of $\langle\alpha, y\rangle \mapsto \alpha$).
- $p_Y : \beta(\kappa \times Y) \to Y$ (the extension of $\langle\alpha, y\rangle \mapsto y$).

The Wallman space of the ultrapower $\prod_u B$ is the fiber $p_\kappa^{\leftarrow}(u)$. Bankston calls this the ultracopower of $Y$; we write $Y_u$.

# Span zero, the real argument

Back to $Z \subseteq wK$.

- Let $Z_u = \mathrm{cl}(\kappa \times Z) \cap p_\kappa^{\leftarrow}(u)$.
- $Z_u$ is a continuum
- $wh[Z_u]$ is a continuum in $(X \times X)_u$ ($wh$ is dual to $h$).
- $Z_X = p_{X\times X}\bigl[wh[Z_u]\bigr]$ is a continuum in $X \times X$.
- And

$$q_K[Z_X] = q_K\Bigl[p_{X\times X}\bigl[wh[Z_u]\bigr]\Bigr] = p_{wK}\Bigl[(wh)^{-1}\bigl[wh[Z_u]\bigr]\Bigr] = Z$$

So, that's it!? Almost.

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# Span zero, the real argument

First expand the language of lattice with two function symbols $\pi_1$ and $\pi_2$.

Apply Shelah's theorem with this extended language. Then $Z_X$ will inherit the mapping properties that $Z$ has.

Finally then: if $X$ is a non-chainable continuum that has span zero (of one of the four kinds) than so is $wL$.

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# Postscript

Logan Hoehn has constructed a metrizable continuum that is non-chainable but that has span zero.

As you all remember from last year's Toposym.

# Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart, B. van der Steeg,
*Span, chainability and the continua* $\mathbb{H}^*$ *and* $\mathbb{I}_u$, Topology and its Applications, 151, 1–3 (2005), 226–237.

D. Bartošová, K. P. Hart, L. Hoehn, B. van der Steeg,
*Lelek's problem is not a metric problem*, Topology and its Applications, 158, Issue 18 (2011), 2479–2484.

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