Applications of the Löwenheim-Skolem theorem. Part II

Non impeditus ab ulla scientia

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Outline

1. Reflections on dimension
   - Dimension functions
   - Formulas
   - Bases
   - Reflections
2. Categoricity
3. Sources

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Covering dimension

Definition (Lebesgue)

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$
(i.e., every $n+2$-element subfamily has an empty intersection).

There is a convenient characterization.

Theorem (Hemmingsen)

$\dim X \leqslant n$ iff every $n+2$-element open cover has a shrinking with an empty intersection.

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Covering dimension

We say $\dim X = n$ if $\dim X \leqslant n$ but $\dim X \nleqslant n-1$;
also, $\dim X = \infty$ means $\dim X \nleqslant n$ for all $n \in \mathbb{N}$.
$\dim X$ is the *covering dimension* of X.

Theorem

$\dim[0,1]^n = n$ *for all* $n \in \mathbb{N} \cup \{\infty\}$.

Thus, dim helps in showing that all cubes are topologically distinct.

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Large inductive dimension

Definition (Čech)

$\operatorname{Ind} X \leqslant n$ if between every two disjoint closed sets A and B there is a partition L that satisfies $\operatorname{Ind} L \leqslant n-1$.
The starting point: $\operatorname{Ind} X \leqslant -1$ iff $X = \emptyset$.

L is a partition between A and B means: there are closed sets F and G that cover X and satisfy: $F \cap B = \emptyset$, $G \cap A = \emptyset$ and $F \cap G = L$.

Large inductive dimension

We say $\operatorname{Ind} X = n$ if $\operatorname{Ind} X \leqslant n$ but $\operatorname{Ind} X \nleqslant n-1$;
also, $\operatorname{Ind} X = \infty$ means $\operatorname{Ind} X \nleqslant n$ for all $n \in \mathbb{N}$.
$\operatorname{Ind} X$ is the *large inductive dimension* of X.

Theorem

$\operatorname{Ind}[0,1]^n = n$ *for all* $n \in \mathbb{N} \cup \{\infty\}$.

Thus, Ind helps in showing that all cubes are topologically distinct.

Dimensionsgrad

Definition (Brouwer)

$\operatorname{Dg} X \leqslant n$ if between every two disjoint closed sets A and B there is a cut C that satisfies $\operatorname{Dg} C \leqslant n-1$.
The starting point: $\operatorname{Dg} X \leqslant -1$ iff $X = \emptyset$.

C is a cut between A and B means: $C \cap K \neq \emptyset$ whenever K is a subcontinuum of X that meets both A and B.

Dimensionsgrad

We say $\operatorname{Dg} X = n$ if $\operatorname{Dg} X \leqslant n$ but $\operatorname{Dg} X \nleqslant n - 1$;
also, $\operatorname{Dg} X = \infty$ means $\operatorname{Dg} X \nleqslant n$ for all $n \in \mathbb{N}$.
$\operatorname{Dg} X$ is the *Dimensionsgrad* of X.

Theorem

$\operatorname{Dg}[0,1]^n = n$ *for all* $n \in \mathbb{N} \cup \{\infty\}$.

Thus, Dg helps in showing that all cubes are topologically distinct.

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Equalities

Theorem

For every compact metrizable space X we have

$$\dim X = \operatorname{Dg} X = \operatorname{Ind} X$$

- $\dim X = \operatorname{Ind} X$ for all metrizable X
- $\dim X = \operatorname{Dg} X$ for all σ-compact metrizable X ...
- ... but not for all separable metrizable X

More inequalities

For compact Hausdorff spaces:

- $\operatorname{Dg} X \leqslant \operatorname{Ind} X$ (each partition is a cut)
- $\dim X \leqslant \operatorname{Ind} X$ (Vedenissof)
- $\dim X \leqslant \operatorname{Dg} X$ (Fedorchuk)

We will (re)prove the last two inequalities algebraically.

Covering dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$ reformulated in terms of closed sets and cast as a formula, δ_n, in the language of lattices

$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2}) \\
&\qquad \big[(x_1 \sqcap x_2 \sqcap \cdots \sqcap x_{n+2} = 0) \to \\
&\qquad \big((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge \cdots \wedge (x_{n+2} \leqslant y_{n+2}) \\
&\qquad \wedge (y_1 \sqcap y_2 \sqcap \cdots \sqcap y_{n+2} = 0) \\
&\qquad \wedge (y_1 \sqcup y_2 \sqcup \cdots \sqcup y_{n+2} = 1)\big)\big].
\end{aligned}
$$

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Large inductive dimension

We can express $\operatorname{Ind} X \leqslant n$ in a similar fashion, the formula $I_n(a)$ becomes (recursively)

$$(\forall x)(\forall y)(\exists u)$$
$$\Big[\big(\big((x \leqslant a) \wedge (y \leqslant a) \wedge (x \sqcap y = \mathfrak{o})\big) \to \big(\operatorname{partn}(u, x, y, a) \wedge I_{n-1}(u)\big)\Big]$$

where $\operatorname{partn}(u, x, y, a)$ says that u is a partition between x and y in the (sub)space a:

$$(\exists f)(\exists g)\big((x \sqcap f = \mathfrak{o}) \wedge (y \sqcap g = \mathfrak{o}) \wedge (f \sqcup g = a) \wedge (f \sqcap g = u)\big).$$

We start with $I_{-1}(a)$, which denotes $a = \mathfrak{o}$

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Dimensionsgrad

Here we have the recursive definition of a formula $\Delta_n(a)$:

$$(\forall x)(\forall y)(\exists u)$$
$$\Big[\big((x \leqslant a) \wedge (y \leqslant a) \wedge (x \sqcap y = \mathfrak{o})\big) \to \big(\mathrm{cut}(u, x, y, a) \wedge \Delta_{n-1}(u)\big)\Big],$$

and $\Delta_{-1}(a)$ denotes $a = \mathfrak{o}$.

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Dimensionsgrad (auxiliary formulas)

The formula $\mathrm{cut}(u, x, y, a)$ expresses that u is a cut between x and y in a:

$$(\forall v)\bigl[\bigl((v \leqslant a) \wedge \mathrm{conn}(v) \wedge (v \sqcap x \neq \mathfrak{o}) \wedge (v \sqcap y \neq \mathfrak{o})\bigr) \rightarrow (v \sqcap u \neq \mathfrak{o})\bigr],$$

and $\mathrm{conn}(a)$ says that a is connected:

$$(\forall x)(\forall y)\bigl[\bigl((x \sqcap y = \mathfrak{o}) \wedge (x \sqcup y = a)\bigr) \rightarrow \bigl((x = \mathfrak{o}) \vee (x = a)\bigr)\bigr],$$

Equivalences

- $\dim X \leqslant n$ iff δ_n holds in 2^X
- $\operatorname{Ind} X \leqslant n$ iff $I_n(X)$ holds in 2^X
- $\operatorname{Dg} X \leqslant n$ iff $\Delta_n(X)$ holds in 2^X

Covering dimension

Theorem

Let X be compact. Then $\dim X \leqslant n$ iff some (every) lattice-base for its closed sets satisfies δ_n.

Proof: compactness and a shrinking-and-swelling argument.

Large inductive dimension

Theorem

Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $I_n(X)$ then $\operatorname{Ind} X \leqslant n$.

Proof: induction and, again, a swelling-and-shrinking argument.

No equivalence, see later.

Dimensionsgrad

Theorem

Let X be compact. If some lattice-base, $\mathcal{B}$, for its closed sets satisfies $\Delta_n(X)$ then we can't say anything about $\operatorname{Dg} X$.

Proof: we can cheat and create, for $[0,1]$ say, a lattice base without connected elements; that base satisfies $\Delta_0(X)$ vacuously.

Take an elementary sublattice

Let X be compact Hausdorff and let $\mathcal{B}$ be a countable elementary sublattice of 2^X.

Let $w\mathcal{B}$ be the ultrafilter space of $\mathcal{B}$;

The w is for Wallman.

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Covering dimension vs large inductive dimension

The formula δ_n holds in $\mathcal{B}$ iff it holds in 2^X, hence

$$\dim w\mathcal{B} = \dim X.$$

The formula $I_n(X)$ holds in $\mathcal{B}$ iff it holds in 2^X, hence

$$\operatorname{Ind} w\mathcal{B} \leqslant \operatorname{Ind} X.$$

But $w\mathcal{B}$ is compact metrizable, so $\dim w\mathcal{B} = \operatorname{Ind} w\mathcal{B}$, hence

$$\dim X \leqslant \operatorname{Ind} X.$$

Covering dimension vs large inductive dimension

There are (many) compact Hausdorff spaces with non-coinciding dimensions, e.g., an early example of a compact L such that $\dim L = 1$ and $\operatorname{Ind} L = 2$ (Lokucievskiĭ).

In that case $\operatorname{Ind} w\mathcal{B} < \operatorname{Ind} L$ for all elementary sublattices of 2^L.

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Covering dimension vs Dimensionsgrad

The stronger inequality $\dim X \leqslant \mathrm{Dg}\, X$ can be proved via $w\mathcal{B}$ as well.
The argument is more involved.
It uses in an essential way that $\mathcal{B}$ is an elementary sublattice of 2^X.

The proof

Let $n = \operatorname{Dg} X$.
Let A and B be closed and disjoint in $w\mathcal{B}$. Wlog: $A, B \in \mathcal{B}$.
Elementarity: there is $C \in \mathcal{B}$ that is a cut between A and B in X and that satisfies $\Delta_{n-1}(C) \leqslant n - 1$.
Inductive assumption: $\operatorname{Dg} C \leqslant n - 1$ in $w\mathcal{B}$, *because* $\mathcal{C} = \{D \in \mathcal{B} : D \subseteq C\}$ is an elementary sublattice of $\{D \in 2^X : D \subseteq C\}$ and C-in-$w\mathcal{B}$ is $w\mathcal{C}$.
Still to show: C-in-$w\mathcal{B}$ is a cut between A and B in $w\mathcal{B}$.

The proof (continued)

Let F be a closed set in $w\mathcal{B}$ that meets A and B but not C.
We show F is not connected.

Find H in $\mathcal{B}$ around F, disjoint from C.

Back in X no component of H meets C, hence it does *not* meet both A and B.

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The proof (continued)

By well-known topology and elementarity there are disjoint elements H_A and H_B of $\mathcal{B}$ such that $H = H_A \cup H_B$, $A \cap H \subseteq H_A$ and $B \cap H \subseteq H_B$.

That well-known topology: the decomposition of H into its components is a zero-dimensional space; hence there is a clopen-in-H set K such that $A \cap H \subseteq K$ and $B \cap H \cap K = \emptyset$.

This yields a formula to apply elementarity to.

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The proof (continued)

Down in $w\mathcal{B}$ we have exactly the same relations, and hence also $F \cap A \subseteq H_A$ and $B \cap F \subseteq H_B$, so H_A and H_B show F is not connected.

Covering dimension vs Dimensionsgrad

The formula δ_n holds in $\mathcal{B}$ iff it holds in 2^X, hence

$$\dim w\mathcal{B} = \dim X.$$

We have shown outright that

$$\operatorname{Dg} w\mathcal{B} \leqslant \operatorname{Ind} X.$$

But $w\mathcal{B}$ is compact metrizable, so $\dim w\mathcal{B} = \operatorname{Dg} w\mathcal{B}$, hence

$$\dim X \leqslant \operatorname{Dg} X.$$

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The result

Given a metric continuum X there is another metric continuum Y such that

- X and Y look the same
  (they have elementarily equivalent countable bases)
- X and Y are not homeomorphic

Example: zero-dimensionality

Here is a first-order sentence, call it ζ

$$(\forall x)(\forall y)(\exists u)(\exists v)$$
$$\big((x \sqcap y = 0) \to ((x \leqslant u) \land (y \leqslant v) \land (u \sqcap v = 0) \land (u \sqcup v = 1))\big)$$

In words: any two disjoint closed sets (x and y) can be separated by clopen sets (u and v).
By *compactness*, if some base satisfies this sentence then the space is zero-dimensional.

Example: no isolated points

Here is a another first-order sentence, call it π

$$(\forall x)(\exists y)\big((x < 1) \to ((x < y) \wedge (y < 1))\big)$$

In words: every closed proper subset (x) is properly contained in a closed proper subset (y);
in fewer words: there are no isolated points.
If some base satisfies this sentence then the space has no isolated points.

Example: the Cantor set is categorical

Let X be compact metric with a countable base $\mathcal{B}$ for the closed sets that satisfies ζ and π.
Then X is zero-dimensional and without isolated points.
So X is (homeomorphic to) the Cantor set C.

Thus: if X looks like C then X is homeomorphic to C.

The Cantor set is categorical among compact metric spaces.

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What the main result says

Among metric continua there is no categorical space.
No (in)finite list of first-order properties will characterize a single metric continuum.

A case in point: the pseudoarc

The pseudoarc is the only metric continuum that is

- hereditarily indecomposable and
- chainable

A two-item list but . . .
Chainability is *not* first-order. (This we will see tomorrow.)
(Hereditary indecomposability is.)

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An embedding lemma

Lemma

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
There is an embedding of $\mathcal{C}$ into the ultrapower of $\mathcal{B}$ by u.

How to make Y

Let X and Z be metric continua, with countable lattice bases, $\mathcal{B}$ and $\mathcal{C}$, for their respective families of closed sets.
Let u be a free ultrafilter on ω.
Let $\varphi : \mathcal{C} \to \mathcal{B}_u$ be an embedding.

Apply the Löwenheim-Skolem theorem:
Find a countable elementary sublattice $\mathcal{D}$ of $\mathcal{B}_u$ that contains $\varphi[\mathcal{C}]$.
Let Y be the Wallman space of $\mathcal{D}$.

Properties of Y

- Y is compact metric ($\mathcal{D}$ is countable).
- $\mathcal{D}$ is a base for the closed sets of Y (by Wallman's theorem).
- $\mathcal{D}$ is elementarily equivalent to $\mathcal{B}_u$ and hence to $\mathcal{B}$.
- Y maps onto Z (because $\varphi[\mathcal{C}]$ is embedded into $\mathcal{D}$).

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Getting a good Y

Let X be given, with a countable base $\mathcal{B}$ for its closed sets.
There is a metric continuum Z that is not a continuous image of X (Waraszkiewicz).
Find Y with a base that is elementarily equivalent to $\mathcal{B}$ and such that Y maps onto Z.
So: Y is not homeomorphic to X.

Light reading

Website: `fa.its.tudelft.nl/~hart`

K. P. Hart.
Elementarity and dimensions, Mathematical Notes, **78** (2005), 264–269.

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There is no categorical metric continuum, Aportaciones Matemáticas, Investigacion **19** (2007), 39–43.

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