RESOLVABILITY, PART 3.

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QUESTION. Are MN spaces maximally resolvable?

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PROBLEM. (Ceder and Pearson, 1967)

Are ω-resolvable spaces maximally resolvable?

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[J-S-Sz] $\equiv$ I. JUHÁSZ, L. SOUKUP AND Z. SZENTMIKLÓSSY, *Resolvability and monotone normality*, Israel J. Math., 166 (2008), no. 1, pp. 1–16.

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– If κ is measurable then there is a MN space X with $\Delta(X) = \kappa$ that is not ω_1-resolvable.

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This left a number of questions open.

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– [Kunen - Prikry, 1971] If $\lambda < \aleph_\omega$ then every $\mathcal{F} \in \mathrm{un}(\lambda)$ is maximally decomposable.

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FACT. [J-S-Sz] Every filtration space $X(F)$ is MN: For $s \in V \in \tau_F$ put

$$H(s,V) = \{t \in V : s \leq t \text{ and } [s,t] \subset V\}$$

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COROLLARY. [J-S-Sz]

If $\mathcal{F} \in \mathrm{un}(\kappa)$ is a measure and $F(t) = \mathcal{F}$ for all $t \in \mathrm{dom}(F) = \kappa^{<\omega}$ then $X(F)$ is hereditarily ω_1-irresolvable.

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Full λ-filtrations were considered in [J-S-Sz].

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reduction results

THEOREM [J-S-Sz]

For $\kappa \leq \lambda = \mathrm{cf}(\lambda)$, TFAEV

– Every DSD space X with $|X| = \Delta(X) = \lambda$ is κ-resolvable.

– Every MN space X with $|X| = \Delta(X) = \lambda$ is κ-resolvable.

– $X(F)$ is κ-resolvable for every full λ-filtration F.

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NOTE. In both results, the case $\kappa = \lambda$ is of main interest.

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Lemma. [J-S-Sz]

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The singular case (proved in [J-M]) is similar but more complicated.

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COROLLARY [J-M]

If every $\mathcal{F} \in \mathrm{un}(\mu)$ is maximally decomposable whenever $\omega \le \mu \le \lambda$, then $X(F)$ is λ-resolvable for any λ-filtration F.