RESOLVABILITY OF TOPOLOGICAL SPACES

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intro

DEFINITION. (Hewitt, 1943, Pearson, 1963)

– A topological space X is κ-resolvable iff it has κ disjoint dense subsets.
– X is maximally resolvable iff it is $\Delta(X)$-resolvable, where

$$\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open in } X\}.$$

FACTS.

– Compact Hausdorff, metric, and linearly ordered spaces are maximally resolvable.

– There is a countable, regular ($\equiv T_3$), dense-in-itself space X (i.e. $|X| = \Delta(X) = \omega$) that is irresolvable ($\equiv$ not 2-resolvable).

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π-networks

LEMMA. (Elkin, 1969)

If $\{Y \subset X : Y \text{ is } \kappa - \text{resolvable}\}$ is a π-network in the space X then X is κ-resolvable.

PROOF. $\{Y_i : i \in I\}$ be a maximal disjoint system of κ-resolvable subspaces of X, $\{D_{i,\alpha} : \alpha < \kappa\}$ be disjoint dense sets in Y_i for $i \in I$. Clearly, then $D_\alpha = \bigcup_{i \in I} D_{i,\alpha}$ for $\alpha < \kappa$ are disjoint dense sets in X.

COROLLARY 1

If every open $G \subset X$ with $|G| = \Delta(G)$ has a κ-resolvable subspace then X is κ-resolvable.

COROLLARY 2

If X is irresolvable then there is an open $Y \subset X$ that is OHI. So, X is irresolvable iff there is an ultrafilter on X generated by open sets.

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DEFINITION. (i) $\mathcal{A}$ is a B_κ-system if $|\mathcal{A}| = \kappa$ and $|A| = \kappa$ for all $A \in \mathcal{A}$.
(ii) X is a B_κ-space if it has a B_κ-system π-network.

THEOREM. Bernstein-Kuratowski

If $\mathcal{A}$ is a B_κ-system then there is a disjoint family $\mathcal{D}$ with $|\mathcal{D}| = \kappa$ s.t. $D \cap A \neq \emptyset$ for all $A \in \mathcal{A}$ and $D \in \mathcal{D}$.

COROLLARY

(i) Any B_κ-space is κ-resolvable.
(ii) Let $\mathcal{C}$ be a class of spaces that is open hereditary and $\pi(X) \leq |X|$ for all $X \in \mathcal{C}$.Then every member of $\mathcal{C}$ is maximally resolvable.

EXAMPLES. Metric spaces, locally compact Hausdorff spaces, GO spaces ($\equiv$ subpaces of LOTS).

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G_δ-resolvable

THEOREM. J-Sz

Every compact T_2 space X is maximally G_δ-resolvable ($\equiv$ X_δ is maximally resolvable).

PROOF. Enough to show: $|X| = \Delta(X_\delta) = \kappa > \omega$ implies $\pi(X_\delta) \le \kappa$.

(i) If $\kappa = \kappa^\omega$ then even $w(X_\delta) \le w(X)^\omega \le \kappa$.

(ii) $\kappa < \kappa^\omega$ let λ be minimal with $\lambda^\omega > \kappa$, then $\mu < \lambda$ implies $\mu^\omega < \lambda$.

$S = \{x \in X : \chi(x, X) < \lambda\}$ is G_δ-dense in X: If $H \subset X$ were closed G_δ with $\chi(x, X) = \chi(x, H) \ge \lambda$ for all $x \in H$ then $|H| \ge 2^\lambda \ge \lambda^\omega > \kappa$ by the Čech-Pospišil thm, contradiction.

But $\chi(x, X_\delta) \le \chi(x, X)^\omega < \lambda \le \kappa$ for $x \in S$, so $\pi(X_\delta) \le \kappa$.

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Every compact T_2 space X is maximally G_δ-resolvable
($\equiv$ X_δ is maximally resolvable).

PROOF. Enough to show: $|X| = \Delta(X_\delta) = \kappa > \omega$ implies $\pi(X_\delta) \le \kappa$.

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DEFINITION. A space is left (right) separated if it has a well-order in which every final (initial) segment is open.

$$ls(X) = \min\{|\mathcal{L}| : X = \cup\mathcal{L} \wedge \forall L \in \mathcal{L} \text{ is left sep'd}\}.$$

LEMMA. (Pavlov, 2002)

If $ls(G) \geq \kappa$ for all open $G \subset X$ then X is κ-resolvable.

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THEOREM. (Pytkeev, 2006)

Every crowded countably compact T_3 space X is ω_1-resolvable.

NOTE. This fails for T_2!

PROOF. Tkachenko (1979): If Y is countably compact T_3 with $Is(Y) \leq \omega$ then Y is scattered. But every open $G \subset X$ includes a regular closed Y, hence $Is(G) \geq Is(Y) \geq \omega_1$.

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Is every crowded countably compact T_3 space X $\mathfrak{c}$-resolvable?

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Assume that $\mathcal{I}$ is a κ-complete ideal on X ($\kappa \geq \omega$) and $D \subset X$ is dense with $|D| \leq \kappa$ s.t. for any $x \in D$ and $Y \in \mathcal{I}$ there is $Z \in \mathcal{I}$ for which $Y \cap Z = \emptyset$ and $x \in \overline{Z}$. Then X is κ-resolvable.

PROOF. Set $D = \{x_\alpha : \alpha < \kappa\}$ s.t. $\forall\, x \in D,\ \ a_x = \{\alpha : x = x_\alpha\} \in [\kappa]^\kappa$. By transf. rec'n on $\alpha < \kappa$ define $Z_\alpha \in \mathcal{I}$ s.t. $(\cup_{\beta<\alpha} Z_\beta) \cap Z_\alpha = \emptyset$ and $x_\alpha \in \overline{Z_\alpha}$. For fixed $i < \kappa$ set

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where $\alpha_{x,i}$ is the ith member of a_x. Then S_i is dense because $D \subset \overline{S_i}$.

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T_κ-points

EXAMPLE. For a regular κ, the ideal $[X]^{<\kappa}$ is κ-complete.

DEFINITION. (i) For κ regular, $x \in X$ is a T_κ-point if for every $Y \in [X]^{<\kappa}$ there is $Z \in [X]^{<\kappa}$ with $Y \cap Z = \emptyset$ and $x \in \overline{Z}$.
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(iii) $T_\kappa(X)$ denotes the set of all T_κ-points of X.

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For any κ, if there is $D \subset T_\kappa(X)$ dense with $|D| \le \kappa$ then X is κ-resolvable.

PROOF (for singular κ). Put $D = \{x_\alpha : \alpha < \kappa\}$ s.t. $\forall\, x \in D$, $\{\alpha : x = x_\alpha\} \in [\kappa]^\kappa$. Define $\{Z_\alpha : \alpha < \kappa\}$ disjoint s.t. $|Z_\alpha| \le \alpha$ and $x_\alpha \in \overline{Z_\alpha}$ if $\mu_{x_\alpha} \le \alpha$. Then use $a_x = \{\alpha : x = x_\alpha \wedge \mu_x \le \alpha\} \in [\kappa]^\kappa$ to obtain the disjoint dense sets $S_i = \cup\{Z_{\alpha_{x,i}} : x \in D\}$, for $i < \kappa$.

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For any κ, if there is $D \subset T_\kappa(X)$ dense with $|D| \le \kappa$ then X is κ-resolvable.

PROOF (for singular κ). Put $D = \{x_\alpha : \alpha < \kappa\}$ s.t. $\forall x \in D$, $\{\alpha : x = x_\alpha\} \in [\kappa]^\kappa$. Define $\{Z_\alpha : \alpha < \kappa\}$ disjoint s.t. $|Z_\alpha| \le \alpha$ and $x_\alpha \in \overline{Z_\alpha}$ if $\mu_{x_\alpha} \le \alpha$. Then use $a_x = \{\alpha : x = x_\alpha \wedge \mu_x \le \alpha\} \in [\kappa]^\kappa$ to obtain the disjoint dense sets $S_i = \cup\{Z_{\alpha_{x,i}} : x \in D\}$, for $i < \kappa$.

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EXAMPLE. For a regular κ, the ideal $[X]^{<\kappa}$ is κ-complete.

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If κ is regular and $T_\kappa(X) = X$ then X is κ-resolvable.

PROOF. $T_\kappa(X) = X$ implies $T_\kappa(Y) = Y$ for all open $Y \subset X$, hence it suffices to show that X has a κ-resolvable subspace.
By regularity of κ, for any $A \in [X]^{<\kappa}$ there is $B \in [X]^{<\kappa}$ s.t. $A \cap B = \emptyset$ and $A \subset \overline{B}$.
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DEFINITION. (i) $p \in X$ is a B_κ-point if there is a B_κ-system (i.e. κ sets of size κ) that forms a local π-network at p.

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EXAMPLES. (i) If the one-one sequence $\{x_\alpha : \alpha < \kappa\}$ converges to p then p is a B_κ-point.
The converse fails, even for $\kappa = \omega$: Any infinite compact T_2 space has a B_ω-point.

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DEFINITION. X is a Pytkeev space if for every non-closed set $A \subset X$ there are a B_κ-point p and $B \in [A]^{\leq\kappa}$ s.t. $p \in \overline{B} \setminus A$.

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THEOREM. (Pytkeev, 1983)

Every Pytkeev space is maximally resolvable.

LEMMA.

Assume that X is a Pytkeev space and $Z \subset X$ is $< \lambda$-closed (i.e. $A \in [Z]^{<\lambda}$ implies $\overline{A} \subset Z$). Then every $y \in \overline{Z} \setminus Z$ is a B_Q-point where $\lambda \leq \min Q$.

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Assume that X is a Pytkeev space and $Z \subset X$ is $< \lambda$-closed (i.e. $A \in [Z]^{<\lambda}$ implies $\overline{A} \subset Z$). Then every $y \in \overline{Z} \setminus Z$ is a B_Q-point where $\lambda \leq \min Q$.

Pytkeev spaces

DEFINITION. X is a Pytkeev space if for every non-closed set $A \subset X$ there are a B_κ-point p and $B \in [A]^{\leq\kappa}$ s.t. $p \in \overline{B} \setminus A$.

EXAMPLES. (i) $B = \{x_\alpha : \alpha < \kappa\} \subset A$ converges to p.

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Assume that $\mu \geq \omega$ and for every $x \in X$ there is some $\kappa \geq \mu$ s.t. x is a B_κ-point. Then X is μ-resolvable.

PROOF. If τ is the topology of X, by Zorn's lemma there is a maximal topology $\varrho \supset \tau$ s.t. if $\mathcal{B}$ witnesses that (for some $\kappa \geq \mu$) x is a B_κ-point w.r.t. τ then the same is true w.r.t. ϱ.

Then $\langle X, \varrho \rangle$ is Pytkeev: If $Y \subset X$ is not ϱ-open then, by maximality, there is a B_κ-point (w.r.t. ϱ) $x \in Y$ and a witness $\mathcal{B}$ for this s.t. $B \setminus Y \neq \emptyset$ for all $B \in \mathcal{B}$. So, there is $Z \in [X \setminus Y]^{\leq \kappa}$ with $x \in \overline{Z}^{\varrho}$.

Thus $\langle X, \varrho \rangle$ is maximally resolvable, while $\Delta(X, \varrho) \geq \mu$, by definition. Consequently, X is μ-resolvable.

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PROOF. If τ is the topology of X, by Zorn's lemma there is a maximal topology $\varrho \supset \tau$ s.t. if $\mathcal{B}$ witnesses that (for some $\kappa \geq \mu$) x is a B_κ-point w.r.t. τ then the same is true w.r.t. ϱ.

Then $\langle X, \varrho \rangle$ is Pytkeev: If $Y \subset X$ is not ϱ-open then, by maximality, there is a B_κ-point (w.r.t. ϱ) $x \in Y$ and a witness $\mathcal{B}$ for this s.t. $B \setminus Y \neq \emptyset$ for all $B \in \mathcal{B}$. So, there is $Z \in [X \setminus Y]^{\leq \kappa}$ with $x \in \overline{Z}^{\varrho}$.

Thus $\langle X, \varrho \rangle$ is maximally resolvable, while $\Delta(X, \varrho) \geq \mu$, by definition. Consequently, X is μ-resolvable.

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