Definable graphs on Polish spaces

Stefan Geschke

January 31, 2012

Cardinal invariants of definable graphs

Definable graphs

Definition

A *graph* is a set X of *vertices* together with a set $E \subseteq [X]^2$ of *edges*.

Equivalently, a graph is a set X together with a symmetric, irreflexive relation R, the *edge relation*.

We usually describe a graph in terms of the set E of edges rather than the edge relation R.

Definition

Let $G = (X, E)$ be a graph. Then $A \subseteq X$ is a *G-clique* (a *clique* in G) if $[A]^2 \subseteq E$.

$A \subseteq X$ is *G-independent* (an *independent* set in G) if $[A]^2 \cap E = \emptyset$. (Independent sets are sometimes called discrete.)

$A \subseteq X$ is *G-homogeneous* (a *homogeneous* set in G) if A is either independent or a clique in G.

Theorem (Ramsey)

Every infinite graph has an infinite homogeneous subset.

This theorem fails if "infinite" is replaced by "uncountable".

Theorem (Sierpiński)

There is a graph on $\mathbb{R}$ whose homogeneous sets are all countable.

Proof.

Let $\sqsubseteq$ be a well-order on $\mathbb{R}$. Two distinct reals x, y form an edge iff $\sqsubseteq$ and $\leq$ agree on the set $\{x, y\}$. $\square$

A version of Ramsey's theorem that yields uncountable homogeneous sets is the Erdős-Rado theorem.

Theorem (Erdős-Rado)

Every graph of size $(2^{\aleph_0})^+$ *has an uncountable homogeneous set.*

However, this theorem does not say anything about graphs on $\mathbb{R}$ or related spaces such as 2^ω and ω^ω.

The proof of Sierpiński's theorem heavily uses the Axiom of Choice. On the other hand, most uncountable graphs occurring in nature are definable in one way or other.

Definition

A graph G on a Hausdorff space X is *closed* (*open*, *clopen*, G_δ, *analytic*, has the *Baire property*) if the edge relation of G as a subset of

$$X^2 \setminus \{(x, x) : x \in X\}$$

is closed (open, clopen, G_δ, analytic, has the Baire property).

A useful uncountable version of Ramsey's theorem holds for sufficiently definable graphs.

Theorem (Galvin; Mycielski)

Let $G = (X, E)$ be a graph with the Baire property on an uncountable Polish space X.
Then there is a perfect G-homogeneous set $H \subseteq X$.

In this context, by *perfect set* we mean a non-empty (!) closed set without isolated points. Perfect subsets of Polish spaces contain a copy of 2^ω and therefore are of size $2^{\aleph_0}$.

Cardinal invariants

As in finite graph theory, we study uncountable graph in terms of their cardinal invariants.

Definition

The *clique-number* of a graph G is the supremum of all sizes of G-cliques.

The *chromatic number* of a graph $G = (X, E)$ is the least size of a family $\mathcal{F}$ of vertices of G such that $X = \bigcup \mathcal{F}$.

The *cochromatic number* (or *homogeneity number*) of a graph $G = (X, E)$ is the least size of a family $\mathcal{F}$ of G-homogeneous sets such that $X = \bigcup \mathcal{F}$.

Why study these cardinal invariants?

- Applications!
- Clique and chromatic numbers are very commonly studied in finite combinatorics. Their duals, independence and clique cover number, are just clique and chromatic number of the complement graph.
- The maximal size of a homogeneous subset of a sufficiently definable graph on an uncountable Polish space is always $2^{\aleph_0}$ by Galvin-Mycielski.
- For graphs of low complexity, the chromatic numbers and clique numbers are degenerate, i.e., either countable or $2^{\aleph_0}$, while the cochromatic number is interesting even for clopen graphs.

Theorem (Todorcevic)

Let $G = (X, E)$ be an open graph on a Polish space X.
Then either G has a perfect clique or X is the union of countably many closed G-independent sets.

This theorem does not hold for closed graphs.

Proof

Suppose G is uncountably chromatic. Wlog assume that no non-empty open subset of X is the union of countably many independent sets. We construct a perfect clique.

Construct a perfect scheme $(U_s)_{s\in 2^{<\omega}}$ of open subsets of X such that

1. if $s \in 2^n$, then $\mathrm{diam}(U_s) < 2^{-n}$,
2. if $s, t \in 2^{<\omega}$ and $s \subsetneq t$, then $\mathrm{cl}(U_t) \subseteq U_s$,
3. if $s \in 2^n$, then $\forall x \in U_{s^\frown 0} \forall y \in U_{s^\frown 1}(\{x, y\} \in E)$.

For every $s \in 2^\omega$ let $f(s)$ be the unique element of $\bigcap_{n\in\omega} U_{s\restriction n}$.

Now $f[2^\omega]$ is a perfect clique in G. $\square$

Theorem (Kubiś)

If a G_δ-graph $G = (X, E)$ on a Polish space X has an uncountable clique, then it has a perfect clique.

This theorem does not hold for F_σ-graphs.

Proof

Let C be an uncountable G-clique and suppose that $E = \bigcap_{n\in\omega} E_n$, the (X, E_n) open graphs.
Wlog assume that for every open set $O \subseteq X$, if $O \cap C \neq \emptyset$, then $O \cap C$ is uncountable.
We construct a perfect G-clique.

Construct a perfect scheme $(U_s)_{s\in 2^{<\omega}}$ of open subsets of X such that

1. if $s \in 2^n$, then $\mathrm{diam}(U_s) < 2^{-n}$ and $U_s \cap C \neq \emptyset$,
2. if $s, t \in 2^{<\omega}$ and $s \subsetneq t$, then $\mathrm{cl}(U_t) \subseteq U_s$,
3. if $s, t \in 2^n$ are distinct, then $\forall x \in U_s \forall y \in U_t(\{x, y\} \in E_n)$.

For every $s \in 2^\omega$ let $f(s)$ be the unique element of $\bigcap_{n\in\omega} U_{s\restriction n}$.

Now $f[2^\omega]$ is a perfect clique in G. $\square$

More systematic approach to cardinal invariants of definable graphs, motivated by set theory:

For a definable graph G on an uncountable Polish space X study σ-ideals on X generated by

- the G-independent sets and
- the homogeneous sets.

Chromatic and cochromatic number are essentially the covering numbers of these σ-ideals.

We might also want to look at the restrictions of these σ-ideals to closed and Borel sets, respectively.

Borel chromatic numbers

The G_0-dichotomy

Definition

Let $G = (X, E)$ be a graph on a Polish space X. The *Borel chromatic number* of G is the least size of a family $\mathcal{F}$ of G-independent Borel sets with $\bigcup \mathcal{F} = X$.

Fact

Note that if G has a clique of size κ, then the chromatic number of G and hence also the Borel chromatic number is at least κ.

Theorem (Kechris, Solecki, Todorcevic)

There is a closed graph G_0 on 2^ω such that for every analytic graph $G = (X, E)$ on a Polish space X, the Borel chromatic number of G is uncountable iff there is a continuous function $f : 2^\omega \to X$ that maps edges of G_0 to edges of G.

Definition

For each $n \in \omega$ choose $s_n \in 2^n$ such that for all $t \in 2^{<\omega}$ there is $n \in \omega$ with $t \subseteq s_n$.
The edges of G_0 are all two-element subsets of 2^ω of the form $\{s_n{}^\frown 0^\frown x, s_n{}^\frown 1^\frown x\}$ where $n \in \omega$, $x \in 2^\omega$, and $^\frown$ denotes concatenation of sequences.

Lemma (KST)

The Borel chromatic number of G_0, and hence of every uncountably Borel chromatic analytic graph, is at least $\mathrm{cov}(\mathtt{meager})$.

Proof.

We show that every G_0-independent set with the Baire property is meager. Let $A \subseteq 2^\omega$ be a G_0-independent set with the Baire property.
Let $O \subseteq 2^\omega$ be open such that $A \triangle O$ meager. For $s \in 2^{<\omega}$ let $[s]$ denote the set of functions in 2^ω extending s. Let $n \in \omega$ be such that $[s_n] \subseteq O$. Now for all $i \in 2$, $[s_n{}^\frown i] \cap A$ is comeager in $[s_n{}^\frown i]$. But since A is G_0-independent, the sets $\{x \in 2^\omega : s_n{}^\frown i^\frown x \in A\}$, $i \in 2$, are disjoint. This contradicts the fact that they are both comeager. $\square$

A forcing dichotomy

Theorem (Conley, G., B. Miller)

Let $G = (\omega^\omega, E)$ be a closed graph. Then either G has a perfect clique or there is a ccc forcing extension of the universe in which ω^ω is covered by $\aleph_1$ compact G-independent sets while $2^{\aleph_0}$ is arbitrarily large.

Corollary

Let X be a Polish space and let $G = (X, E)$ be a closed graph. Then either G has a perfect clique or there is a ccc forcing extension of the universe in which the Borel chromatic number of G is $\aleph_1$ while $2^{\aleph_0}$ is arbitrarily large.

Sketch of the proof of the CGM theorem

For every closed graph $G = (\omega^\omega, E)$ let $\mathbb{P}(G)$ denote the partial order defined as follows:

A pair $q = (T_p, F_p)$ is a condition in $\mathbb{P}(G)$ if

1. T_p is a finite subtree of $\omega^{<\omega}$,
2. there is some $m_p \in \omega$ such that all maximal elements of T_p are of length m_p,
3. if s and t are two distinct maximal elements of T_p, then for all $x, y \in \omega^\omega$ with $s \subseteq x$ and $t \subseteq y$ we have $\{x, y\} \notin E$,
4. F_p is a finite subset of ω^ω, and
5. for all $\{x, y\} \in [F_p]^2$, $x \restriction m_p \in T_p$ and $x \restriction m_p \neq y \restriction m_p$.

For two conditions $p, q \in \mathbb{P}(G)$ we let $p \leq q$ iff

6. $T_q \subseteq T_p$ and T_q consists precisely of all elements of T_p of length $\leq m_q$.
7. $F_q \subseteq F_p$.

This forcing adds a generic compact G-independent set.

A finite support iteration of $\mathbb{P}(G)$ of countable length adds a countable family of compact G-independent sets that covers the ground model elements of ω^ω.

Lemma

The forcing notion $\mathbb{P}(G)$ is ccc iff G has no perfect cliques.

The proof of this lemma relies on Kubiś's theorem that G_δ-graphs with uncountable cliques have perfect cliques and on the Galvin-Mycielski theorem about the existence of perfect homogeneous sets in Borel graphs.

If G has no perfect cliques, we can use the lemma to show that a finite support iteration of $\mathbb{P}(G)$ of length ω_1 yields a model of set theory in which the weak Borel chromatic number of G is $\aleph_1$. Note that for absoluteness reasons no perfect G-cliques can be added by forcing. $\square$

Using some book keeping and forcings of the type $\mathbb{P}(G)$, we can obtain a model in which for each closed graph without a perfect clique, the forcing adding $\aleph_1$ compact independent sets covering all vertices has already been carried out.

Theorem

It is consistent with an arbitrarily large size of the continuum that every closed graph G on a Polish space X has either a perfect clique or the Borel chromatic number of G is $\aleph_1$.

Compare this to the theorem on open colorings by Todorcevic.

Proof.

Force with a finite support product of various $\mathbb{P}(G)$'s. Iterate ω_1 times. □

Cochromatic numbers

Theorem (G., Kojman, Kubiś, Schipperus)

There is a clopen graph G_{min} on 2^ω such that every clopen graph G on a Polish space has an uncountable cochromatic number iff G_{min} embeds into G.

Theorem (G.)

The cochromatic number of G_{min}, and hence of every uncountably cochromatic clopen graph on a Polish space, is at least $\mathrm{cof}(\texttt{null})$.

Theorem (G., Kojman, Kubiś, Schipperus)

It is consistent that every clopen graph on a Polish space has cochromatic number at most $\aleph_1 < 2^{\aleph_0}$.

This theorem does not hold for open or closed graphs.

The proof strategy is completely different from proof of our consistency result about chromatic numbers:

Start with a model of GCH and iterate Sacks forcing with countable supports of length ω_2.

This yields a model of $2^{\aleph_0} = \aleph_2$ and for every clopen graph $G = (X, E)$ added at stage $\alpha < \omega_2$, the X in the final model is covered by the G-homogeneous sets in the α-th intermediate model.

An example

Definition

Fix a homeomorphism $h : 2^\omega \to (2^\omega)^\omega$.

For each $n \in \omega$ let $\pi_n : (2^\omega)^\omega \to 2^\omega$ be the projection to the n-th coordinate and $f_n = \pi_n \circ h$.

For distinct $x, y \in 2^\omega$ let $\{x, y\} \in E$ iff there is $n \in \omega$ such that $f_n(x) = y$ or $f_n(y) = x$.

Now $G = (2^\omega, E)$ is graph on the Cantor space that is F_σ since the edge-relation is

$$R = \bigcup_{n\in\omega} (f_n \cup f_n^{-1}) \setminus \{(x, x) : x \in 2^\omega\}$$

Theorem

G has an uncountable clique, but no perfect clique.

Proof.

Every maximal G-clique is uncountable.
If $P \subseteq 2^\omega$ is a perfect clique, then P is homeomorphic to 2^ω and

$$P^2 \subseteq \mathrm{id}_P \cup \bigcup_{n\in\omega} (f_n \cup f_n^{-1}).$$

But the sets id_P and $(f_n \cup f_n^{-1}) \cap P$ are nowhere dense in P^2 and hence do not cover P^2. □

Theorem (Mátrai)

The chromatic number of G is $\aleph_1$.

Proof.

Let F be the closure of the set $\{\mathrm{id}_{2^\omega}\} \cup \{f_n : n \in \omega\}$ under composition. For $x, y \in 2^\omega$ let $y \leq x$ iff there is $f \in F$ such that $f(x) = y$. Observe that for each $x \in 2^\omega$, $\{y \in 2^\omega : y \leq x\}$ is countable.

If $A \subseteq 2^\omega$ is G-independent, then the downward closure $\mathrm{cl}_\leq(A)$ of A with respect to $\leq$ is the union of not more than $\aleph_1$ independent sets.

Choose a sequence $(A_\alpha)_{\alpha<\omega_1}$ of subsets of 2^ω such that each A_α is a maximal G-independent subset of $2^\omega \setminus \bigcup_{\nu<\alpha} \mathrm{cl}_\leq(A_\nu)$.

Now $2^\omega = \bigcup_{\nu<\omega_1} A_\nu$. □

Theorem (G)

The Borel chromatic number of G is at least $\operatorname{cov}(\mathtt{meager})$ *and equal to* $\aleph_1$ *in a ccc forcing extension of the set-theoretic universe with an arbitrarily large* $2^{\aleph_0}$*.*

Proof.

Note that by Kubiś's theorem on G_δ-graphs with uncountable cliques, every F_σ-graph with an uncountable independent set has a perfect independent set.

Let $A \subseteq 2^\omega$ be G-independent. There is a ccc forcing notion that generically adds countably many compact G-independent sets that cover A.

Now cover 2^ω by $\aleph_1$ G-independent sets. Force to cover each of the $\aleph_1$ G-independent sets by countably many compact G-independent sets. Iterate this of length ω_1. $\square$

Clopen graphs and their finite induced subgraphs

"Only two things are infinite, the universe and human stupidity, and I'm not shure about the former."

Albert Einstein

Modular profinite graphs

Definition

Let $G = (V(G), E(G))$ be a graph. $M \subseteq V(G)$ is a *module* of G if for all $u, v \in M$ and all $w \in V(G) \setminus M$ we have

$$\{u, w\} \in E(G) \quad \Leftrightarrow \quad \{v, w\} \in E(G).$$

A partition of $V(G)$ into modules is a *modular partition* of G.

For graphs F and G a map $f : V(F) \to V(G)$ is *modular* if $\{f^{-1}(w) : w \in V(G)\}$ is a modular partition of F and for all $u, v \in V(F)$ with $f(u) \neq f(v)$,

$$\{u, v\} \in E(F) \quad \Leftrightarrow \quad \{f(u), f(v)\} \in E(G).$$

Definition

Let $(I, \leq)$ be a directed (partially ordered) set. For each $i \in I$ let $G_i = (V_i, E_i)$ be a graph. For $i, j \in I$ with $i \leq j$ let $\pi^j_i : V_j \to V_i$ be a modular map from G_j to G_i.

Then $((G_i)_{i \in I}, (\pi^j_i)_{i,j \in I, i \leq j})$ is an *inverse system* of graphs (with modular bonding maps) iff

1. for all $i \in I$, π^i_i is the indentity on G_i and
2. for all $i, j, k \in I$ with $i \leq j \leq k$, $\pi^k_i = \pi^j_i \circ \pi^k_j$.

Definition

Let $((G_i)_{i\in I},(\pi^j_i)_{i,j\in I,i\leq j})$ be an inverse system of graphs with modular bonding maps.
A graph $G=(V,E)$ together with a family $(\pi_i)_{i\in I}$ of maps is the *limit* of the inverse system $((G_i)_{i\in I},(\pi^j_i)_{i,j\in I,i\leq j})$ if

1. for all $i\in I$, π_i is a modular map from G to G_i,
2. for all $i,j\in I$ with $i\leq j$, $\pi_i=\pi^j_i\circ\pi_j$,
3. if G' is a graph and $(\pi'_i)_{i\in I}$ is a family of maps satisfying 1. and 2., then there is a unique modular map π from G' to G such that for all $i\in I$, $\pi'_i=\pi_i\circ\pi$.

Definition

A graph G is *modular profinite* if it is the limit of an inverse system of finite graphs with modular bonding maps.

Remark

Since every modular profinite graph is the limit of an inverse system of finite sets, it carries a compact zero-dimensional topology. Every modular profinite graph is clopen with respect to this topology.

Theorem

A graph G is modular profinite iff $V(G)$ is compact and any two distinct vertices v and w are separated by a finite, modular partition of G consisting of clopen sets.

Definition

For distinct $x, y \in \omega^\omega$ let

$$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}.$$

Let $X \subseteq \omega^\omega$ and $c : [X]^2 \to 2$. Then the coloring c is of *depth n* if for all distinct $x, y \in X$, $c(x, y)$ only depends on $x \restriction \Delta(x, y) + n$ and $y \restriction \Delta(x, y) + n$.
A clopen graph on a subset X of ω^ω is of depth n if it is of the form $(X, c^{-1}(1))$ for some coloring $c : [X]^2 \to 2$ of depth n.

Theorem

A graph G on a topological space $V(G)$ of countable weight is modular profinite iff it is (homeomorphically) isomorphic to a graph of depth 1 on a compact subset of ω^ω.

Example

For distinct $x, y \in 2^\omega$ let $n = \Delta(x, y)$ and

$$c(x,y) = \begin{cases} 0, & \text{if } x(n+1) \neq y(n+1) \text{ and} \\ 1, & \text{if } x(n+1) = y(n+1). \end{cases}$$

Then the graph $(2^\omega, c^{-1}(1))$ is *prime*, i.e., it has no non-trivial modules. In particular, the graph is not modular profinite. Also, the graph has an induced subgraph that is an infinite path.

Observe that an infinite path is prime and that a modular profinite graph has only finite induced subgraphs that are prime.

Finite induced subgraphs of clopen graphs

Definition

For a graph G let $\mathrm{age}(G)$ be the *age* of G, the class of all finite graphs isomorphic to induced subgraphs of G.

If $V(G)$ is a topological space, let $\mathrm{hage}(G)$ denote the *hereditary age of* G, the class of all finite graphs that have induced copies in every nonempty open subset of $V(G)$.

For a class $\mathcal{C}$ of finite graphs let $\mathrm{cl}(\mathcal{C})$ denote the closure of $\mathcal{C}$ under isomorphic copies, induced subgraphs, and substitution of graphs for vertices.

A class of finite graphs is *finitely generated* if it is the closure of a finite set of finite graphs.

Lemma

a) For every graph G on a topological space $V(G)$, $\mathrm{hage}(G)$ *is closed under substitution and therefore*

$$\mathrm{hage}(G) = \mathrm{cl}(\mathrm{hage}(G)).$$

b) For every class $\mathcal{C}$ of finite graphs there is a modular profinite graph $G_{\mathcal{C}}$ of countable weight such that

$$\mathrm{age}(G_{\mathcal{C}}) = \mathrm{hage}(G_{\mathcal{C}}) = \mathrm{cl}(\mathcal{C}).$$

$G_{\mathcal{C}}$ is unique up to bi-embeddability.

Lemma

Let F be a modular profinite graph of countable weight and let G be a clopen graph on a Polish space.
If $\operatorname{age}(F) \subseteq \operatorname{hage}(G)$, *then F embeds into G homeomorphically.*

Definition

A graph G on a topological space $V(G)$ is *self-similar* if $\operatorname{age}(G) = \operatorname{hage}(G)$.

Lemma

The embeddability relation between self-similar modular profinite graphs of countable weight is bi-reducible with $\subseteq$ on $\mathcal{P}(\omega)$.

Example

Classes of finite graphs closed under substitution, isomorphic copies, and induced subgraphs are the class of P_4-free graphs, which is generated by an edge and a non-edge, the class of perfect graphs, and the class of all finite graphs.

We call the corresponding modular profinite graphs of countable weight G_{min}, G_{perf}, and G_{max}.

By the previous results, G_{min} embeds into G_{perf} and G_{perf} embeds into G_{max}, but not vice versa.

Cochromatic numbers revisited

Definition

Let $\mathfrak{hm}(G)$ denote the cochromatic number of the graph G.

Recall that $\mathfrak{hm}(G_{\mathrm{min}}) \geq \mathrm{cof}(\mathtt{null}) \geq \mathfrak{d}$ and that every Polish space is the union of not more than $\mathfrak{d}$ Cantor spaces and singletons.

Lemma (G., Goldstern, Kojman)

a) Every clopen graph on a compact, zero-dimensional, metric space is homeomorphically isomorphic to a graph of depth 2 on a compact subset of ω^ω.
b) For every graph G of depth 2 on a closed subset X of ω^ω, X is the union of no more than $\mathfrak{hm}$ compact sets Y such that $G \restriction Y$ is of depth 1, i.e., modular profinite.

Theorem

Let G be a clopen graph on a Polish space.

a) H is the union of no more than $\mathfrak{hm}(G_{\mathrm{min}})$ induced subgraphs that embed into $G_{\mathrm{age}(H)}$ and hence into G_{max}. In particular, $\mathfrak{hm}(G) \leq \mathfrak{hm}(G_{\mathrm{max}})$. (G., Goldstern, Kojman)

b) $\mathfrak{hm}(G_{\mathrm{perf}}) < \mathfrak{hm}(G_{\mathrm{max}})$ is consistent. (G., Goldstern, Kojman)

c) If H is self-similar, then $\mathfrak{hm}(H) = \mathfrak{hm}(G_{\mathrm{age}(H)})$.

d) If $\mathrm{age}(H)$ is finitely generated, then $\mathfrak{hm}(H) \leq \mathfrak{hm}(G_{\mathrm{min}})$.

Conclusion

$$\mathrm{cof}(\mathtt{null}) \leq \mathfrak{hm}(G_{\mathrm{min}}) \leq \mathfrak{hm}(G_{\mathrm{max}}) \leq 2^{\aleph_0} \leq \mathfrak{hm}(G_{\mathrm{min}})^+$$

All the inequalities are consistently strict, at least individually.

- S. Geschke, *Clopen graphs, inverse limits, and cochromatic numbers*, preprint
- S. Geschke, *Weak Borel chromatic numbers*, Mathematical Logic Quarterly, Volume 57, Issue 1 (2011), 5–13
- S. Geschke, M. Goldstern, M. Kojman, *Continuous Ramsey theory on Polish spaces and covering the plane by functions*, Journal of Mathematical Logic, Vol. 4 (2004), No. 2, 109–145
- S. Geschke, M. Kojman, W. Kubiś, R. Schipperus, *Convex decompositions in the plane and continuous pair colorings of the irrationals*, Israel J. Math. 131 (2002), 285–317
- A. Kechris, S. Solecki, S. Todorčević, Borel chromatic numbers, Adv. Math. 141 (1999), 1–44

Thank you!