Coanalytic Transfinite Constructions

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2-point sets

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Theorem. (Bouhjar, Dijkstra, and van Mill) It cannot be F_σ!

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Inductive proof

Standard proof of the existence:

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Question

The set of possible choices is very large. Can we construct a "nice" set?

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Irregularity

- Con(exists an A $\mathbf{\Pi}^1_1$ set such that $\omega < |A| < 2^\omega$)
- Con($\exists$ an uncountable coanalytic set without a perfect subset)

Miller's theorem

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Method

Miller's method is frequently needed, but he does not give a general condition. The proof is hard, uses effective descriptive set theory and model theory.

General method

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Theorem 1. (V=L) Let $F \subset \mathbb{R}^{\leq\omega} \times \mathbb{R} \times \mathbb{R}$ be a coanalytic set. Assume that for every $(A, p) \in \mathbb{R}^{\leq\omega} \times \mathbb{R}$ the section $F_{(A,p)}$ is Turing-cofinal.

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Then there exists an enumeration of $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$ and a coanalytic set $X = \{x_\alpha : \alpha < \omega_1\}$, such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(\{x_n : n \in \omega\}, p_\alpha)})$, where $\{x_n ; n \in \omega\}$ is a certain enumeration of $\{x_\beta : \beta < \alpha\}$.

Effectiveness

$\Sigma^0_1(y)$, $\Pi^0_1(y)$

Definition. Let $\{I_n : n \in \omega\}$ be a recursive enumeration of the open intervals with rational endpoints. An open set G is called *recursive in* y, iff there exists a subsequence $\{n_k : k \in \omega\} \leq_T y$, such that $G = \cup_k I_{n_k}$. (the class of these sets is denoted by $\Sigma^0_1(y)$).

$$\Pi^0_1(y) = \{G^c : G \in \Sigma^0_1(y)\}$$

We can define these classes similarly for subsets of ω, $\omega \times \mathbb{R}$, $\mathbb{R}^2$ etc. using a recursive enumeration of $\{n\}$, $\{n\} \times I_m$, $I_n \times I_m$ etc.

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The lightface classes

Let us define for $n \geq 2$

$$\Sigma^0_n(y) = \{proj_{\mathbb{R}}(A) : A \subset \mathbb{R} \times \omega, A \in \Pi^0_{n-1}(y)\},$$

$$\Pi^0_n(y) = \{A^c : A \in \Sigma^0_n(y)\}.$$

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Lightface and boldface

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Key theorem

For $x, y \subset \omega$ if $x \in \Delta^1_1(y)$ then x is called *hyperarithmetic* in y, denoted by $x \leq_h y$.

Theorem. (Spector, Gandy) Suppose that a set $A \subset \mathbb{R}^2$ is coanalytic. Then $(\exists y \leq_h x)((x, y) \in A)$ is also coanalytic.

Strengthening

Cofinality in hyperdegrees

Definition. A set $X \subset \mathbb{R}$ is called *cofinal in hyperdegrees* if $(\forall z \in \mathbb{R})(\exists y \in X)(z \leq_h y)$.

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Stronger version

Theorem 2. (V=L) Let $y \in \mathbb{R}$, $F \subset \mathbb{R}^{\leq \omega} \times \mathbb{R} \times \mathbb{R}$ be a $\Pi^1_1(y)$ set. Assume that for every $(A, p) \in \mathbb{R}^{\leq \omega} \times \mathbb{R}$ the section $F_{(A,p)}$ is cofinal in hyperdegrees. Then there exists an enumeration of $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$ and a $\Pi^1_1(y)$ set $X = \{x_\alpha : \alpha < \omega_1\}$, such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(\{x_n : n \in \omega\}, p_\alpha)})$, where $\{x_n : n \in \omega\}$ is a certain enumeration of $\{x_\beta : \beta < \alpha\}$.

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Remark

The previous theorem holds true replacing $\mathbb{R}$ with $\mathbb{R}^n$, ω^ω or 2^ω.

Consequences

Miller's results

Theorem 1. implies Miller's results: consistent existence of coanalytic MAD family, 2-point set and Hamel basis.

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Miller's results

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V=L?

If the condition holds then $\omega_1^L = \omega_1$. Is it equivalent to $(2^\omega)^L = 2^\omega$?

Consequences: C^1 curves

Existence

(CH) There exists an uncountable $X \subset \mathbb{R}^2$ intersecting every C^1 curve in countably many points.

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Theorem 1. implies that under (V=L) there exists an uncountable coanalytic $X \subset \mathbb{R}^2$ set intersecting every C^1 curve in countably many points.

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Remark

In almost every case there are no $\mathbf{\Sigma}^1_1$ sets.

Thank you!