Spaces not mappable onto [0, 1]

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Hausdorff topological space - $X, Y, \ldots$

Definitions

J. Haleš, 2005
A topological space X is an **nCM-space** (**n**on-**C**ontinuously **M**appable space) if X cannot be continuously mapped onto [0,1].

- J. Isbell, 1965, 1969; A.W. Miller, 1983

A topological space X is an **nBM-space** (**n**on-**B**orel **M**appable space) if X cannot be mapped onto [0,1] by any Borel map.

A uniform space X is an **nUCM-space** (**n**on-**U**niformly **C**ontinuously **M**appable space) if X cannot be uniformly continuously mapped onto [0,1].

- P. Corazza, 1989

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- preserved by appropriate maps

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$$\text{non}(\text{nBM-space}) = \text{non}(\text{nCM-space}) = \text{non}(\text{nUCM-space}) = \mathfrak{c}$$

Theorem (Miller, 1983)

The theory **ZFC** $+ \mathfrak{c} = \aleph_2 + (\forall A \subseteq {}^{\omega}2)(A$ *is an* nCM*-set* $\equiv |A| < \mathfrak{c})$ *is consistent relative to* **ZFC**.

Corollary (Corazza, 1989)

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- $\alpha < \mathfrak{c}$: $\{x_\xi;\ \xi < \alpha\}$, $\langle P_\xi : \xi < \alpha \rangle$
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nBM-set of cardinality $\mathfrak{c}$ - **CH**, **MA**, $\mathfrak{p} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{c}$, **MA(countable)**, ...

Corollary

The following statements are undecidable in **ZFC**.

"there exists an nBM*-set of cardinality* $\mathfrak{c}$*"*
"there exists an nCM*-set of cardinality* $\mathfrak{c}$*"*
"there exists an nUCM*-set of cardinality* $\mathfrak{c}$*"*

Theorem (folklore)

ind(X)$=0$ *for any completely regular (Tychonoff)* nCM*-space* X.
Ind(Y)$=0$ *for any normal* nCM*-space* Y.

Theorem

ind(X)$=0$ *for any uniform* nUCM*-space* X.

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Any separable metrizable nUCM*-space is homeomorphic to a subset of* ${}^{\omega}2$.

- any second-countable topological space X is metrizable if and only if X is regular

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ind(X)$=0$ for any uniform nUCM-space X.

Theorem (Isbell, 1965)

For an nUCM*-space* $A \subseteq {}^{\omega}2$ *there is a perfect set* $P \subseteq {}^{\omega}2 \setminus A$*.*

A subset A of a perfect Polish space X is called Marczewski null measurable ((s^0)-set) if any perfect subset of X contains a perfect subset disjoint with A.

Corollary (Corazza, 1989)

An nUCM*-subset* A *of a perfect Polish space* X *is Marczewski null measurable.*

Metric separable space X is totally imperfect if X does not contain a homeomorphic copy of the perfect Cantor set ${}^{\omega}2$.

$$\begin{array}{ccccccccc} \text{nBM} & \longrightarrow & \text{nCM} & \longrightarrow & \text{nUCM} & \longrightarrow & (s^0) & \longrightarrow & (\text{TI}) \\ & & & & \downarrow & & & & \\ & & & & \text{0-dim} & & & & \end{array}$$

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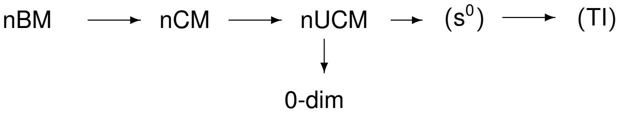

Theorem

The following statements are equivalent.

1. X *is an* nCM*-space.*
2. $[0,1] \setminus f(X)$ *is dense in* $[0,1]$ *for any continuous* $f: X \to [0,1]$.
3. $f(X)$ *is zero-dimensional for any continuous* $f: X \to [0,1]$.
4. $f(X)$ *is totally imperfect for any continuous* $f: X \to [0,1]$.
5. $f(X)$ *is Marczewski null measurable for any continuous* $f: X \to [0,1]$.
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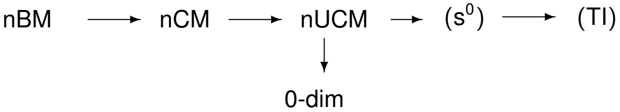

Let $\mathcal{P}$ be a topological property. X is **projectively $\mathcal{P}$** if every continuous image of X into perfect Polish space is $\mathcal{P}$.

Theorem

X is an nCM-space if and only if X is projectively nCM-space.

Corollary

The following statements are equivalent.

1. *X is an nCM-space.*
2. *X is projectively zero-dimensional.*
3. *X is projectively totally imperfect.*
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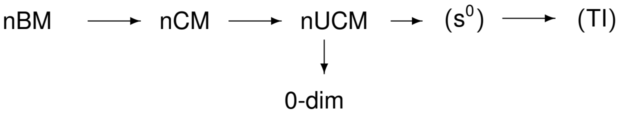

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nBM $\longrightarrow$ nCM $\longrightarrow$ nUCM $\longrightarrow$ (s^0) $\longrightarrow$ (TI)

nUCM $\downarrow$ 0-dim

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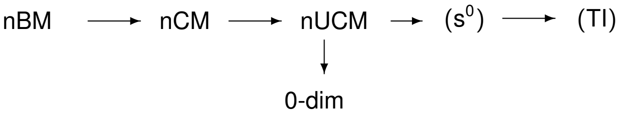
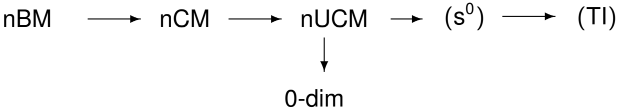


$$\mathrm{cov}(\mathcal{P}\text{-set}) = \min\{|\mathcal{A}|;\ \bigcup\mathcal{A} = [0,1] \wedge (\forall A \in \mathcal{A})\ \text{“}A \text{ is a } \mathcal{P}\text{-set”}\}$$

$$\mathrm{add}(\mathcal{P}\text{-set}) = \min\{|\mathcal{A}|;\ \text{“}\bigcup\mathcal{A} \text{ is not a } \mathcal{P}\text{-set”} \wedge (\forall A \in \mathcal{A})(\ A \subseteq [0,1] \wedge \text{“}A \text{ is a } \mathcal{P}\text{-set”})\}$$

$$\mathrm{add}(\mathcal{P}\text{-space}) = \min\{|\mathcal{A}|;\ (\exists X)\text{“}X \text{ is a topological (uniform) space”} \wedge \mathcal{A} \subseteq \mathcal{P}(X)$$
$$\wedge (\forall A \in \mathcal{A})\ \text{“}A \text{ is a } \mathcal{P}\text{-space”} \wedge \text{“}\bigcup\mathcal{A} \text{ is not a } \mathcal{P}\text{-space”}\}$$



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$$\mathrm{cov}((s^0)\text{-set}) \le \mathrm{cov}(\text{nUCM-set}) \le \mathrm{cov}(\text{nCM-set}) \le \mathrm{cov}(\text{nBM-set})$$

Corollary (Isbell, 1965; Corazza, 1989)

$\aleph_1 \le \mathrm{add}(\text{nCM-space})$, $\aleph_1 \le \mathrm{add}(\text{nUCM-space})$, $\mathrm{add}(\text{nBM-space}) \le \mathfrak{c}$.

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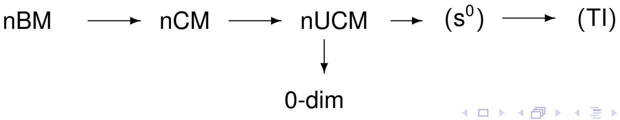

$$\text{add(nBM-space)} = \text{add(nBM-set)} = \text{cov(nBM-set)}$$
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Corollary (Isbell, 1965; Corazza, 1989)

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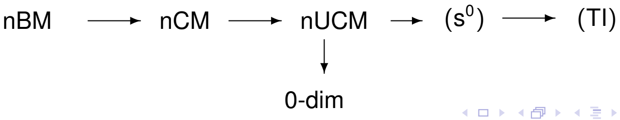

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$$\text{nBM} \longrightarrow \text{nCM} \longrightarrow \text{nUCM} \longrightarrow (\mathrm{s}^0) \longrightarrow (\text{TI})$$
$$\text{nUCM} \downarrow \text{0-dim}$$

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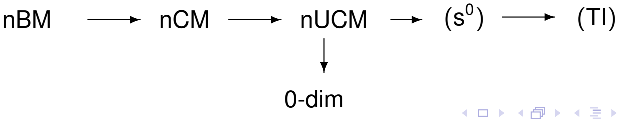
Theorem (Miller, 1983)

ZFC $+ \mathfrak{c} = \kappa + \mathrm{add}(\text{nCM-space}) = \aleph_1$ *is consistent relative to* **ZFC** *(*κ *is a cardinal such that* $\mathrm{cf}(\kappa) > \aleph_0$*).*



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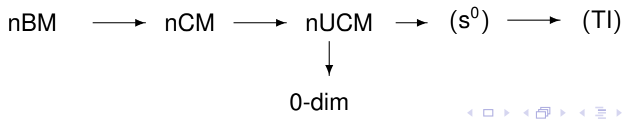
Theorem

ZFC $+ \mathfrak{c} = \aleph_2 + \text{add}(\text{nCM-space}) = \aleph_2$ *is consistent relative to* **ZFC**.



Theorem

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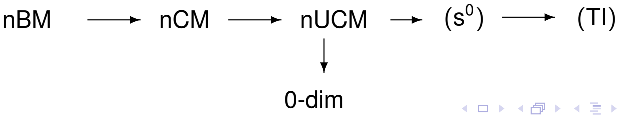
A topological space X is **hereditarily nCM-space**, shortly hnCM-space, if any subset of X is an nCM-space.

Theorem (Corazza, 1989)

If **CH** *holds true then there is an* nCM*-set which is not hereditarily* nCM*-set.*



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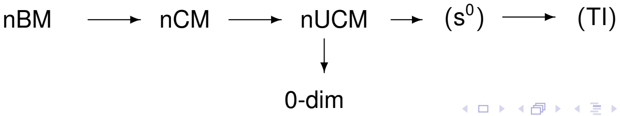

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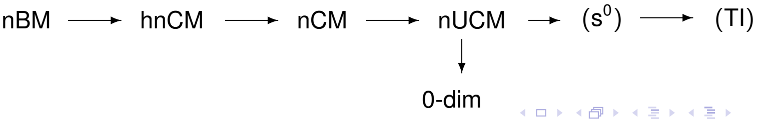

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κ an uncountable cardinal not greater than $\mathfrak{c}$; X a Polish space; $\mathcal{I}$ a σ-additive ideal which has Borel base and $\bigcup \mathcal{I} = X$.

A subset $L \subseteq X$ is called a **κ-$\mathcal{I}$-set** if $|L| \geq \kappa$ and $|L \cap A| < \kappa$ for any $A \in \mathcal{I}$.

- κ-$\mathcal{N}$-set - **κ-Sierpiński** set
- κ-$\mathcal{M}$-set - **κ-Luzin** set
- $\aleph_1$-Luzin set - **Luzin** set
- $\aleph_1$-Sierpiński set - **Sierpiński** set.

κ-$\mathcal{I}$-set of cardinality κ can be constructed under the assumption $\kappa = \text{cov}(\mathcal{I}) = \text{cof}(\mathcal{I})$.

There is $\mathfrak{c}$-Sierpiński set if and only if $\text{cov}(\mathcal{N}) = \mathfrak{c}$.

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An ideal I of a Boolean algebra B is said to be κ-saturated if every subset $A \subseteq B \setminus I$ such that $a \wedge b \in I$ for any $a, b \in A$, $a \neq b$ has cardinality $|A| < \kappa$.

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Theorem (Miller, 1983)

Let $\mathcal{I} \subseteq$ *Borel*$(\mathbb{R})$ *be* $\mathfrak{c}$*-saturated ideal of Borel*$(\mathbb{R})$. *Any* $\mathfrak{c}$-$\mathcal{I}$*-set* A *is an* nBM*-set.*

a topological space X; $\mathcal{G} \subseteq \mathcal{P}(X)$

$$\mathcal{G}_0 = \mathcal{G} \subseteq \mathcal{G}_1 = \mathcal{G}_\sigma \subseteq \dots \subseteq \mathcal{G}_\alpha \subseteq \dots$$

- $\mathcal{G}_{\omega_1} = \mathcal{G}_{\omega_1+1}$
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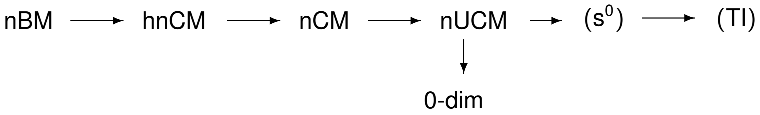

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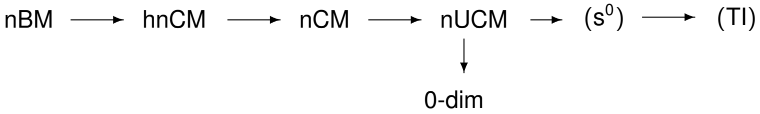
Theorem (Reclaw, 1989?)

Let X be a separable metric space. If X is of bounded Borel rank then X is an nBM-space. In particular, any σ-set is an nBM-space.



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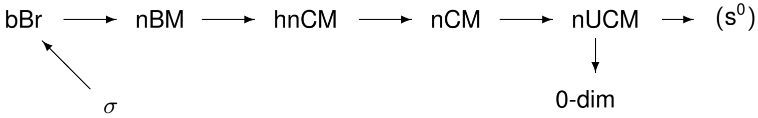
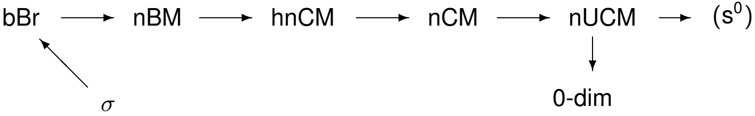


Theorem (Szpilrajn-Marczewski, 1930)

An $\aleph_1$-Sierpiński set is a σ-space.



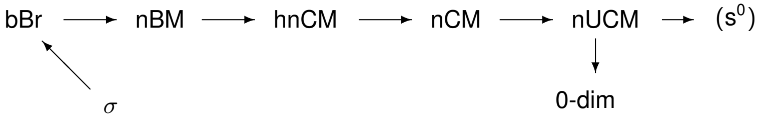
γ **-cover** $\mathcal{U}$ - every $x \in X$ lies in all but finitely many members of $\mathcal{U}$ and $X \notin \mathcal{U}$

$S_1(\Gamma, \Gamma)$-property, 1996

For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of countable open γ-covers, there exist sets $U_n \in \mathcal{U}_n$ such that $\{U_n;\ n \in \omega\}$ is an open γ-cover.

Theorem (Haleš, 2005)

Perfectly normal space X is hereditarily $S_1(\Gamma, \Gamma)$-space if and only if X is both, an $S_1(\Gamma, \Gamma)$-space and a σ-space.



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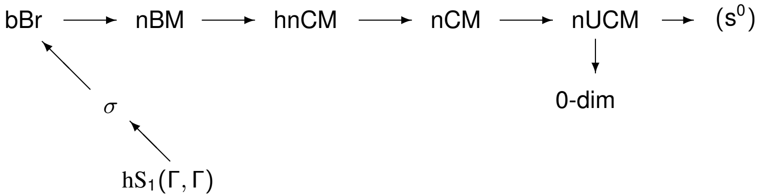
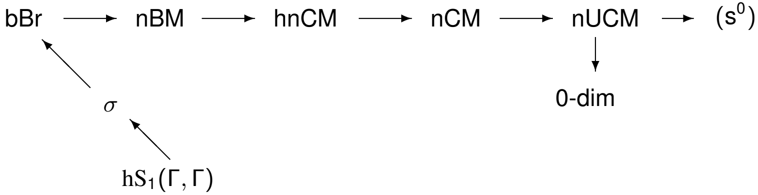


$f_n \xrightarrow{\text{QN}} f$ on X

1. if there exists a sequence of positive reals $\langle \varepsilon_n : n \in \omega \rangle$ converging to zero
2. for any $x \in X$:

$$|f_n(x) - f(x)| < \varepsilon_n$$

holds for all but finitely many $n \in \omega$

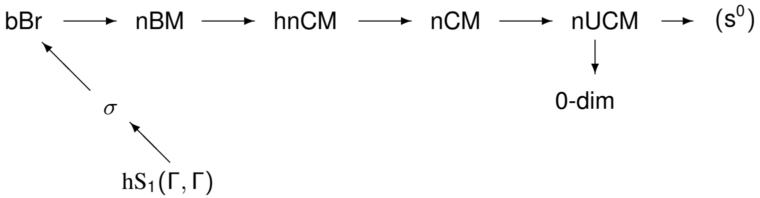


QN-property, 1991

X has the property QN if each sequence of continuous real valued functions converging pointwise to zero is converging to zero quasi-normally.

Theorem (Reclaw, 1997)

Any metric QN-space is a σ-space.



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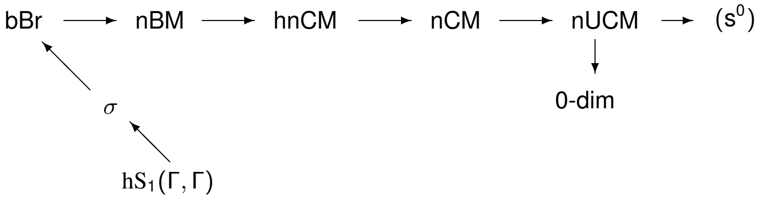
Perfectly normal QN-space is an $S_1(\Gamma, \Gamma)$-space.



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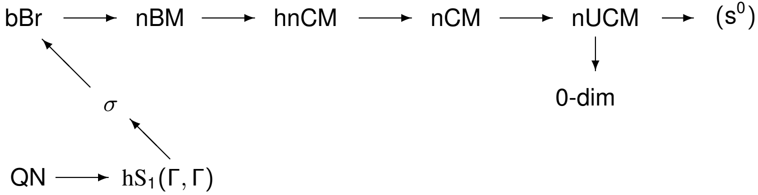
Theorem (Reclaw, 1997)

Any metric QN-space is a σ-space.

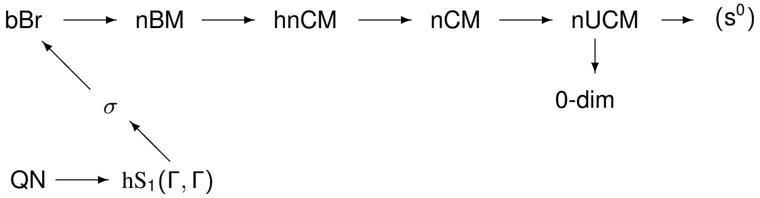
Theorem (Bukovský–Haleš, Sakai, 2007)

Perfectly normal QN-space is an $S_1(\Gamma, \Gamma)$-space.



Theorem (L. Bukovský, I. Reclaw, M. Repický, 1991)

$\mathfrak{b}$-Sierpinski set is a QN-set.



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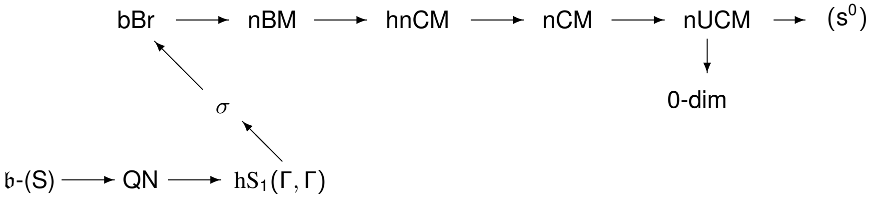
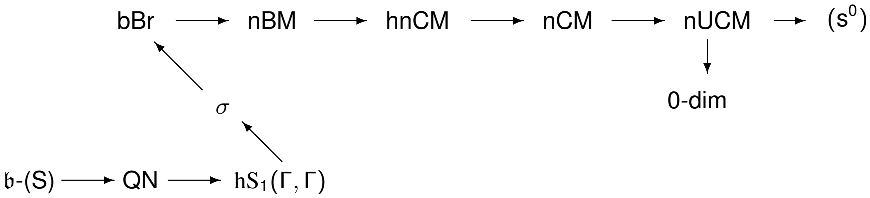


Theorem (Poprougénko, Sierpiński, 1930)

An $\aleph_1$-Lusin set has bounded Borel rank.



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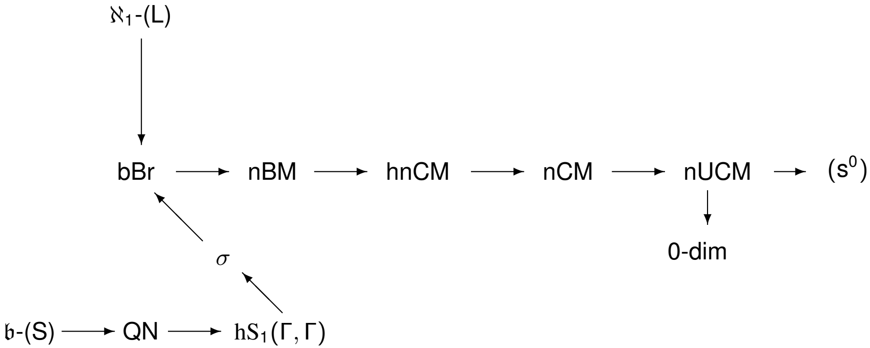

Theorem

Let $\mathcal{P}$ be a topological property such that the unit interval $[0, 1]$ is not $\mathcal{P}$. If X is projectively $\mathcal{P}$ then X is an nCM*-space.*

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wQN-property (weak QN), 1991

X has the property wQN if each sequence of continuous real valued functions converging to zero has a subsequence converging to zero quasi-normally.

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Theorem (Bukovský, Reclaw, Repický, 1991)

Any wQN-space is an nCM-space.

Let X be a Polish space. A set $A \subseteq X$ has strongly measure zero if for any sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of positive real numbers there is a sequence $\langle A_n : n \in \omega \rangle$ of open sets such that $A \subseteq \bigcup_{n\in\omega} A_n$ and $\operatorname{diam}(A_n) < \varepsilon_n$ for any $n \in \omega$.

Lemma

Let X, Y be Polish spaces, $A \subseteq X$ and $f : A \to Y$. If f is uniformly continuous and A has strong measure zero then $f(A)$ has strong measure zero as well.

Theorem (Corazza, 1989)

Any subset of a Polish space X with strong measure zero is an nUCM-space.

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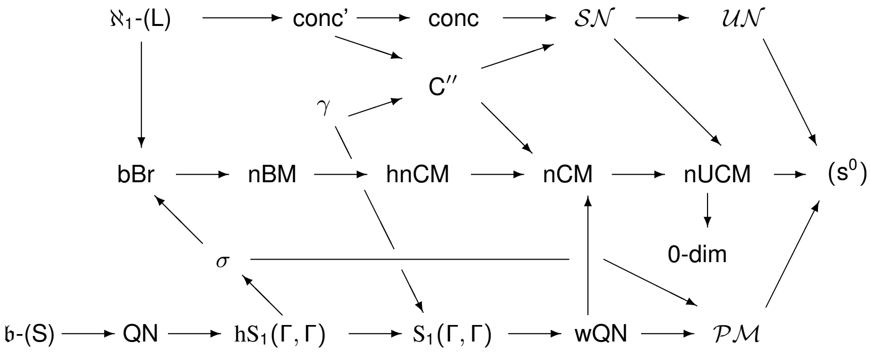

C''	Rothberger property $S_1(\mathcal{O},\mathcal{O})$	$\mathcal{SN}$	strong measure zero
γ	property γ	$\mathcal{UN}$	universal measure zero
conc	concentrated	$\mathcal{PM}$	perfectly meager
conc'	concentrated on its subset		

Theorem (Just–Miller–Scheepers–Szeptycki, 1996)

If $\mathfrak{t} = \mathfrak{b}$ then there exists a set of reals $X \subseteq {}^{\omega}2$ such that X is an $\mathrm{S}_1(\Gamma, \Gamma)^*$-space and X is not σ-compact.

Theorem (Scheepers, 1999)

A topological space X is an $\mathrm{S}_1(\Gamma, \Gamma)^*$-space if and only if X is an $\mathrm{S}_1(\Gamma, \Gamma)$-space

Theorem (see e.g. Bukovský, 2011)

If $\mathfrak{t} = \mathfrak{b}$ then there exists a set of reals $X \subseteq {}^{\omega}2$ of cardinality $\mathfrak{b}$ such that X is an $\mathrm{S}_1(\Gamma, \Gamma)$-space and $X \setminus [\omega]^{\omega}$ is not a wQN-space. Hence, X is not a λ-set.

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If $\mathfrak{t} = \mathfrak{c}$ then there exists a set of reals $X \subseteq {}^{\omega}2$ of cardinality $\mathfrak{c}$ such that X is an $\mathrm{S}_1(\Gamma, \Gamma)$-space and $X \setminus [\omega]^{\omega}$ is not an nCM-space.

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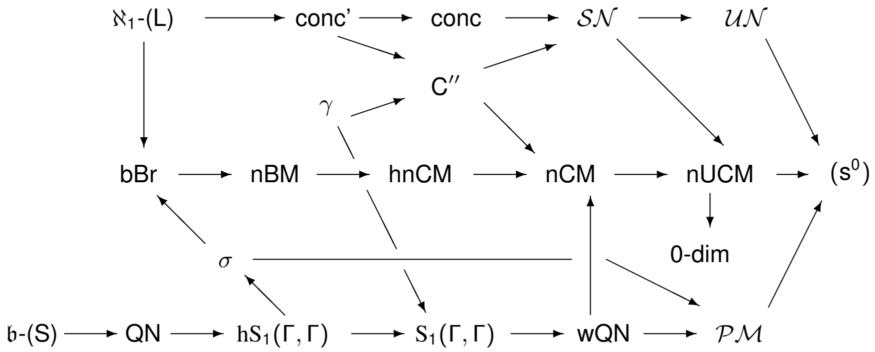

C''	Rothberger property $S_1(\mathcal{O}, \mathcal{O})$	$\mathcal{SN}$	strong measure zero
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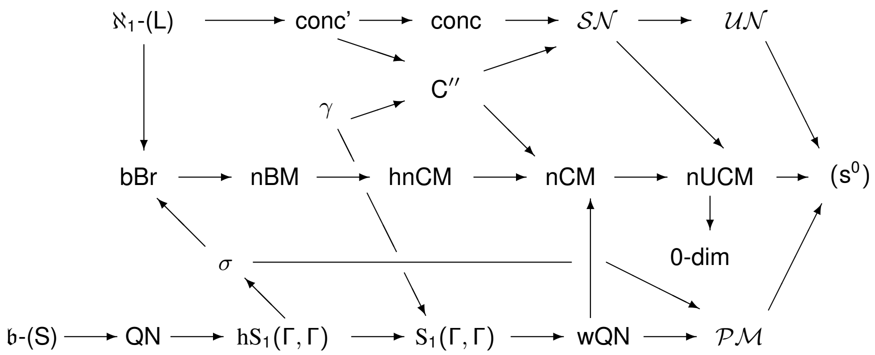
A subset A of a topological space X is called perfectly meager if for any perfect set $P \subseteq X$ the intersection $A \cap P$ is meager in the subspace P.

A subset A of a perfectly normal topological space X has universal measure zero if for any finite diffused Borel measure μ on X we have $\mu^*(A) = 0$, i.e. $\mu(B) = 0$ for some Borel B such that $A \subseteq B$.

Theorem (Miller, 1983; Corazza, 1989)

There is a model of **ZFC** *such that* $\mathfrak{c} = \aleph_2$ *and the following holds:*

- *the following statements are equivalent for* $A \subseteq \mathbb{R}$
  1. $|A| < \mathfrak{c}$.
  2. *A is an* nCM*-set.*
  3. *A is hereditarily* nCM*-set.*
  4. *A is an* nBM*-set.*
  5. *A is an* nUCM*-set.*
- *any perfectly meager set is an* nBM*-set.*
- *there is an* nBM*-set which is not perfectly meager set.*
- *any universal measure zero set is an* nBM*-set.*
- *there is an* nBM*-set which is not universal measure zero set.*



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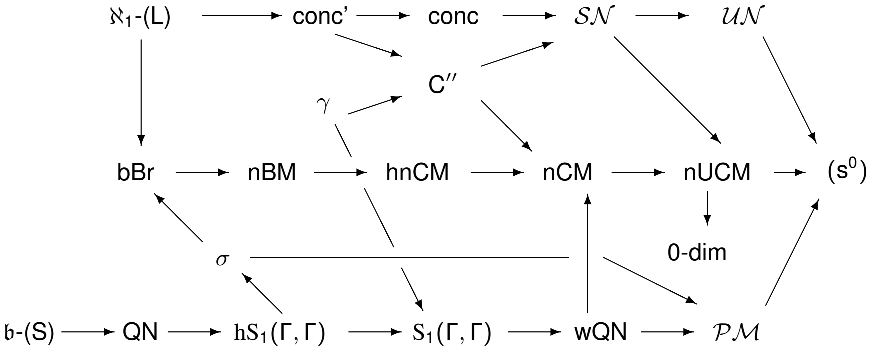
  5. *A is an* nUCM*-set.*
  6. *A has strong measure zero (i.e. Generalized Borel Conjecture holds).*
- *any* nUCM*-set has universal measure zero.*
- *there is universally measure zero set of cardinality* $\mathfrak{c}$ *(which is not an* nUCM*-set).*



C''	Rothberger property $\mathrm{S}_1(\mathcal{O}, \mathcal{O})$	$\mathcal{SN}$	strong measure zero
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Theorem (Ciesielski, Shelah, 1999)

There is a model of **ZFC** *such that* $\mathfrak{c} = \aleph_2$ *and the following holds:*

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  2. *A is an* nCM*-set.*
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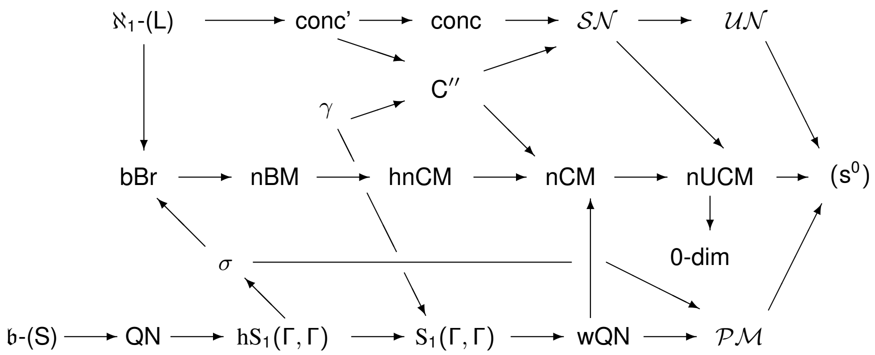
  4. *A is an* nBM*-set.*
  5. *A is an* nUCM*-set.*
- *any* nCM*-set is perfectly meager.*
- *there is perfectly meager set of cardinality* $\mathfrak{c}$ *(which is not an* nCM*-set).*



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Theorem (Corazza, 1989)

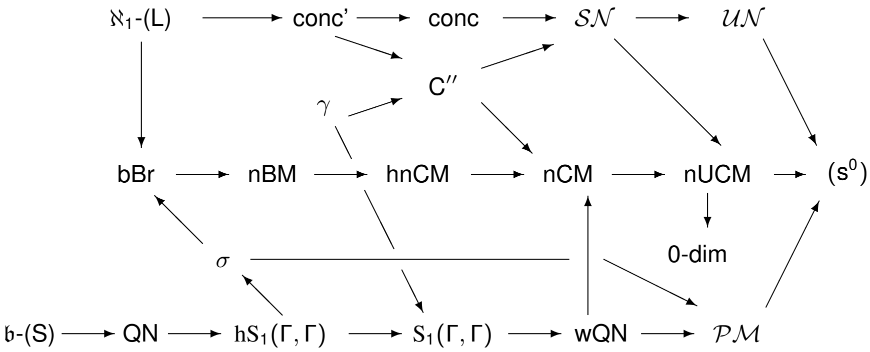
There is Marczewski null measurable set $X \subseteq {}^{\omega}2$ which is not an nUCM*-space.*

Theorem (Hilgers, 1937)

Any separable metric space of cardinality $\mathfrak{c}$ is a continuous injective image of a separable metric spaces of every non-negative dimension including infinite dimension.

Corollary (Mazurkiewicz, Szpilrajn-Marczewski, 1937)

1. *If there is a λ-set (separable metric) of cardinality $\mathfrak{c}$ (e.g. if non$(\mathcal{M}) = \mathfrak{c}$ or if $\mathfrak{b} = \mathfrak{c}$) then there is a λ-set of any dimension.*
2. *If there is a universal measure zero set of cardinality $\mathfrak{c}$ (e.g. if non$(\mathcal{N}) = \mathfrak{c}$) then there is a universal measure zero set of any dimension.*



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Theorem (Miller, 1979)

The theory **ZFC** $+$ *"any uncountable set of reals has unbounded Borel rank" is consistent relative to* **ZFC**.

Theorem (Bukovský, Reclaw, Repický, 1991)

$\mathrm{non}(\text{wQN-space}) = \mathfrak{b}$.

Theorem (Rothberger, 1941)

If **CH** *holds true then there is concentrated set which is not* nCM*-set.*

Theorem (Corazza, 1989)

If **CH** *holds true then there is concentrated set on its subset which is not hereditarily* nCM*-set.*

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Thanks for Your attention!