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Independent Bernstein sets and algebraic constructions

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Introduction

Background

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The notion of algebrability has its origin in works of Aron, Pérez-García and Seoane-Sepulveda and the following is a slightly simplified version of their definition.

**Definition (Aron, Pérez-García and Seoane-Sepulveda)**

Let $\mathcal{L}$ be an algebra. A set $A \subseteq \mathcal{L}$ is said to be $\beta$-algebrable if there exists an algebra $\mathcal{B}$ so that $\mathcal{B} \subseteq A \cup \{0\}$ and $\text{card}(Z) = \beta$, where $\beta$ is cardinal number and $Z$ is a minimal system of generators of $\mathcal{B}$. Here, by $Z = \{z_\alpha : \alpha \in \Lambda\}$ is a minimal system of generators of $\mathcal{B}$, we mean that $\mathcal{B} = A(Z)$ is the algebra generated by $Z$, and for every $\alpha_0 \in \Lambda$, $z_{\alpha_0} \notin A(Z \{z_{\alpha_0}\})$. We also say that $A$ is algebrable if $A$ is $\beta$-algebrable for $\beta$-infinite.
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We study the following classes of functions:

- Perfectly everywhere surjective (PES), strongly everywhere surjective (SES) and everywhere discontinuous Darboux (EDD) functions;
- Everywhere discontinuous functions that have finitely many values (EDF) and everywhere discontinuous compact to compact functions (EDC);
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Independent family of sets

Let $B$ be a family of subsets of a set $X$. We say that the family $A$ is $B$-independent iff

$$A_{\varepsilon_1}^1 \cap \ldots \cap A_{\varepsilon_n}^n \in B$$

for any distinct $A_i \in A$, any $\varepsilon_i \in \{0, 1\}$ for $i \in \{1, \ldots, n\}$ and $n \in \mathbb{N}$ where $A^0 = X \setminus A$ and $A^1 = A$.

There is an independent family of $2^\kappa$ many subsets of $\kappa$. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be a decomposition of $\mathbb{R}$ into disjoint Bernstein sets.

Let $\{N_\xi : \xi < 2^\mathfrak{c}\}$ be an independent family in $\mathfrak{c}$ such that for every $\xi_1 < \ldots < \xi_n < 2^\mathfrak{c}$ and for any $\varepsilon_i \in \{0, 1\}$ the set $N_{\xi_1}^{\varepsilon_1} \cap \ldots \cap N_{\xi_n}^{\varepsilon_n}$ is nonempty and has cardinality $\mathfrak{c}$. 
Independent family of sets

Let $B$ be a family of subsets of a set $X$. We say that the family $A$ is $B$-independent iff

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Let $\{B_\alpha : \alpha < c\}$ be a decomposition of $\mathbb{R}$ into disjoint Bernstein sets.

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For $\xi < 2^c$ put

$$B^\xi = \bigcup_{\alpha \in \mathbb{N}^\xi} B_\alpha.$$ 

Then every set $B^\xi$ is Bernstein. Note that for every $\xi_1 < \ldots < \xi_n < 2^c$ and any $\varepsilon_i \in \{0, 1\}$ the set

$$(B^{\xi_1})^{\varepsilon_1} \cap \ldots \cap (B^{\xi_n})^{\varepsilon_n} = \bigcup_{\alpha \in \mathbb{N}^{\xi_1} \cap \ldots \cap \mathbb{N}^{\xi_n}} B_\alpha$$

is a Bernstein. That means \(\{B^\xi : \xi < 2^c\}\) is the independent family of Bernstein sets.
Let for $\alpha < c$, $g_\alpha : B_\alpha \to \mathbb{C}$ (or $\mathbb{R}$) be a non-zero function. Let us put

$$f_\xi(x) = \begin{cases} g_\alpha(x) & \text{, when } x \in B_\alpha \text{ and } \alpha \in \mathbb{N}_\xi \\ 0 & \text{otherwise.} \end{cases}$$

Then the family $\{f_\xi : \xi < 2^c\}$ is linearly independent.
Remark

Let $P$ be any non-zero polynomial without constant term and consider the function $P(f_{\xi_1}, ..., f_{\xi_n})$. Let

$$P_s(x) = P(\varepsilon_1 \cdot x, ..., \varepsilon_n \cdot x), s = (\varepsilon_1, ..., \varepsilon_n)$$

Let us observe here that the function $P(f_{\xi_1}, ..., f_{\xi_n})|_{B_\alpha}$ for any $\alpha \in \mathcal{N}_{\xi_1} \cap ... \cap \mathcal{N}_{\xi_n}$ is of the form

$$P(\varepsilon_1 \cdot g_\alpha, ..., \varepsilon_n \cdot g_\alpha) = P_s(g_\alpha)$$
Remark

Then we have two possibilities.

(i) Either at least one of the functions $P_s(x)$ for $s \in \{0, 1\}^n$ is a non-zero polynomial of one variable. If $P_s$ is non-zero, where $s = (\varepsilon_1, \ldots, \varepsilon_n)$, then the function $P(f_{\xi_1}, \ldots, f_{\xi_n})$ is non-zero on the Bernstein set of the form

$$(B_{\xi_1})^{\varepsilon_1} \cap (B_{\xi_2})^{\varepsilon_2} \cap \ldots \cap (B_{\xi_n})^{\varepsilon_n}.$$ 

(ii) Or every function of a type $P_s(x)$ is a zero function, and then $P(f_{\xi_1}, \ldots, f_{\xi_n})$ is zero function.

Span the algebra by the functions $\{f_{\xi} : \xi < 2^c\}$ and we get an algebra of $2^c$ many generators.
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Span the algebra by the functions $\{ f_{\xi} : \xi < 2^\xi \}$ and we get an algebra of $2^\xi$ many generators.
\( \mathbb{K} \) is \( \mathbb{R} \) or \( \mathbb{C} \). The function \( f : \mathbb{K} \rightarrow \mathbb{K} \) is called:

- perfectly everywhere surjective (\( \mathcal{PES}(\mathbb{K}) \)) iff for every perfect set \( P \subseteq \mathbb{K} \), \( f(P) = \mathbb{K} \);
- strongly everywhere surjective (\( \mathcal{SES}(\mathbb{K}) \)) iff it takes every real or complex value \( c \) times on any interval.

The real function is an everywhere discontinuous Darboux function (\( \mathcal{EDD}(\mathbb{R}) \)) iff it is nowhere continuous and maps connected sets to connected sets.

**Proposition**

Let \( B \subseteq \mathbb{K} \) be a Bernstein set. There exist a function \( f \in \mathcal{PES}(\mathbb{K}) \) that is 0 on the set \( B^0 \).
K is R or C. The function \( f : K \to K \) is called:

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proof (Sketch)

Let $B \subseteq \mathbb{K}$ be a Bernstein set and $\{P_\alpha : \alpha < c\}$ an enumeration of all perfect sets in $\mathbb{K}$ and $\mathbb{K} = \{y_\beta : \beta < c\}$.

Then for every $\alpha < c$ cardinality of $B_\alpha = P_\alpha \cap B$ is continuum. Ennumerate a product $\{B_\alpha : \alpha < c\} \times \{y_\beta : \beta < c\}$ as $\{A_\gamma : \gamma < c\}$, where $A_\gamma = (B_\gamma, y_\gamma)$.

Choose $x_0 \in B_0$ and put $f(x_0) = y_0$.

Assume that for some $\zeta < c$ the points $\{x_\eta : \eta < \zeta\}$ were chosen satisfying $x_\eta \in B_\eta \setminus \{x_\xi : \xi < c\}$ for every $\eta < \zeta$ with $f(x_\eta) = y_\eta$ for every $\eta < \zeta$.

Put $X = \{x_\eta : \eta < \zeta\}$ then $|X| < c$. So there exists a point $x_\zeta \in B_\zeta \setminus X$ and define $f(x_\zeta) = y_\zeta$. By putting $f(x) = 0$ for every $x \in \mathbb{K} \setminus \{x_\xi : \xi < c\}$ we are done.
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The following theorems hold and the proof is using a family of independent Bernstein sets.

Theorem
The set $PES(\mathbb{C})$ is $2^c$-algebrable.

Theorem
The set $SES(\mathbb{C}) \setminus PES(\mathbb{C})$ is $2^c$-algebrable.

Theorem
The set $EDD(\mathbb{R})$ is $2^c$-algebrable.
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The following theorems hold and the proof is using a family of independent Bernstein sets.

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The set $\mathcal{PES}(\mathbb{C})$ is $2^c$-algebrable.

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The set $\mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C})$ is $2^c$-algebrable.

**Theorem**
The set $\mathcal{EDD}(\mathbb{R})$ is $2^c$-algebrable.
The set $\mathcal{EDF}(\mathbb{R})$ is the set of all nowhere continuous real functions with $|f(\mathbb{R})| < \omega$.

$\mathcal{EDC}(\mathbb{R})$ is the set of all nowhere continuous compact-to-compact functions.

**Theorem**

The set $\mathcal{EDF}(\mathbb{R})$ is $2^\mathfrak{c}$-algebrable but it is not strongly $1$-algebrable.

**Corollary**

The set $\mathcal{EDC}(\mathbb{R})$ is $2^\mathfrak{c}$-algebrable.
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**Theorem**

The set $\mathcal{EDF}(\mathbb{R})$ is $2^\mathfrak{c}$-algebrable but it is not strongly $1$-algebrable.

**Corollary**

The set $\mathcal{EDC}(\mathbb{R})$ is $2^\mathfrak{c}$-algebrable.
Let $C \subsetneq \mathbb{R}$ be a fixed closed subset of $\mathbb{R}$. We consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of $C$.

**Theorem**

The set of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of $C$ is $2^c$-algebraizable.
Let $C \subseteq \mathbb{R}$ be a fixed closed subset of $\mathbb{R}$. We consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of $C$.

**Theorem**

The set of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of $C$ is $2^c$-algebrable.
proof (Sketch)

Let \([1, 2] = \{ r_\alpha : \alpha < c \}\) and 
\(g : \mathbb{R} \to \mathbb{R}\) be such that \(g(x) = d(x, C)\). Then \(g\) is zero only on the set \(C\).

Put \(g_\alpha(x) = r_\alpha \cdot g(x)\) and \(f_\xi\) as in the general method.

If each function \(P_s(x)\) is zero then \(P(f_{\xi_1}, \ldots, f_{\xi_n})\) is zero function.
If \(P_{s_0}(x)\) is non-zero for some \(s_0 \in \{0, 1\}^n\). Then \(P(f_{\xi_1}, \ldots, f_{\xi_n})\) is continuous in any point of \(C\) and suppose that is continuous in a point \(x_0 \not\in C\).
proof (Sketch)

Let \([1, 2] = \{r_\alpha : \alpha < c\}\) and \(g : \mathbb{R} \to \mathbb{R}\) be such that \(g(x) = d(x, C)\). Then \(g\) is zero only on the set \(C\).

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$P(f_{\xi_1}, \ldots, f_{\xi_n})$ is zero on the Bernstein set

$$\bigcup_{\alpha \in N_0^{\xi_1} \cap N_0^{\xi_2} \cap \ldots \cap N_0^{\xi_n}} B_\alpha.$$ 

For every $\beta \in N_0^{\xi_1} \cap N_0^{\xi_2} \cap \ldots \cap N_0^{\xi_n}$ there exist a sequence $(x_n)_{n \in \mathbb{N}} \subseteq B_\beta$ such that $x_n \to x_0$. Hence by the continuity of polynomial of one variable we get that $P_{s_0}(g_\beta(x_0)) = 0$ for any such $\beta$.

Since for $\alpha \neq \beta$ we have that $g_\alpha(x_0) = r_\alpha \cdot g(x_0) \neq r_\beta \cdot g(x_0) = g_\beta(x_0)$ so $P_{s_0}(g_\beta(x_0))$ as a polynomial of one variable $\beta$, that has infinitely many zeros, is zero function - contradiction.
$P(f_{\xi_1}, \ldots, f_{\xi_n})$ is zero on the Bernstein set

$$\bigcup_{\alpha \in N_{\xi_1}^0 \cap N_{\xi_2}^0 \cap \ldots \cap N_{\xi_n}^0} B_\alpha.$$ 

For every $\beta \in N_{\xi_1}^{\varepsilon_1} \cap N_{\xi_2}^{\varepsilon_2} \cap \ldots \cap N_{\xi_n}^{\varepsilon_n}$ there exist a sequence $(x_n)_{n \in \mathbb{N}} \subseteq B_\beta$ such that $x_n \to x_0$. Hence by the continuity of polynomial of one variable we get that $P_{s_0}(g_\beta(x_0)) = 0$ for any such $\beta$.

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Proof continued

\[ P(f_{\xi_1}, \ldots, f_{\xi_n}) \] is zero on the Bernstein set

\[ \bigcup_{\alpha \in N_{\xi_1}^0 \cap N_{\xi_2}^0 \cap \ldots \cap N_{\xi_n}^0} B_\alpha. \]

For every \( \beta \in N_{\xi_1}^{\varepsilon_1} \cap N_{\xi_2}^{\varepsilon_2} \cap \ldots \cap N_{\xi_n}^{\varepsilon_n} \) there exist a sequence \( (x_n)_{n \in \mathbb{N}} \subseteq B_\beta \) such that \( x_n \to x_0 \). Hence by the continuity of polynomial of one variable we get that \( P_{s_0}(g_\beta(x_0)) = 0 \) for any such \( \beta \).

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so \( P_{s_0}(g_\beta(x_0)) \) as a polynomial of one variable \( \beta \), that has infinitely many zeros, is zero function - contradiction.
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Question 2
Is there a function $f \in \mathcal{EDC}(\mathbb{R})$ that has infinitely many values on each interval?

Question 3
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Thank you for your attention :}