Strong algebrability of series and sequences

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Introduction

algebrability

Assume that B is a linear algebra, that is, a linear space being also an algebra. E is κ–algebrable if $E \cup \{0\}$ contains a κ-generated algebra, i.e.
$P(x_1, \ldots, x_n) \in E$ or $P(x_1, \ldots, x_n) = 0$ for distinct generators $x_1, \ldots, x_n$ and any polynomials P.

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free linear algebras

A is a κ-generated free algebra, if there exists a subset $X = \{x_\alpha : \alpha < \kappa\}$ of A such that any function f from X to some algebra A', can be uniquely extended to a homomorphism from A into A'. A subset $X = \{x_\alpha : \alpha < \kappa\}$ of a commutative algebra B generates a free sub-algebra A if and only if for each polynomial P and any $x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_n}$ we have $P(x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_n}) = 0$ if and only if $P = 0$.

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A subset E of a commutative linear algebra B is *strongly κ–algebrable*, if there exists a κ-generated free algebra A contained in $E \cup \{0\}$.

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Proposition

The set c_{00} is ω–algebrable in c_0 but is not strongly 1–algebrable.

Theorem

The set $c_0 \setminus \bigcup\{l^p : p \geq 1\}$ is densely strongly $\mathfrak{c}$–algebrable in c_0.

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The set of all sequences in l^∞ which set of limits points is homeomorphic to the Cantor set is comeager and strongly $\mathfrak{c}$-algebrable.

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A function $f : \mathbb{R} \to \mathbb{R}$ such that, for any set $Z \subset \mathbb{R}$ of cardinality the continuum, the restriction $f|_Z$ is not a Borel map is called Sierpiński-Zygmund function.

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The set of Sierpiński-Zygmund functions is strongly κ-algebrable, provided there exists a family of κ almost disjoint subsets of $\mathfrak{c}$.

lemma

Let $\mathcal{P}$ be a family of non-zero real polynomials with no constant term and let X be a subset of $\mathbb{R}$ both of cardinality less than $\mathfrak{c}$. Then there exists set $Y = \{y_\xi : \xi < \mathfrak{c}\}$ such that $P(y_{\xi_1}, y_{\xi_2}, \ldots, y_{\xi_n}) \notin X$ for any n, any polynomial $P \in \mathcal{P}$ and any distinct ordinals $\xi_i < \mathfrak{c}$.

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Proof. Enumerate Borel functions $\{g_\alpha : \alpha < \mathfrak{c}\}$ and $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$. Let $\{P_\alpha : \alpha < \mathfrak{c}\}$ denote non-zero polynomials without constant term. At the stage α we use Lemma for $X := \{g_\lambda(x_\alpha) : \lambda \leq \alpha\}$ and $\mathcal{P} := \{P_\beta : \beta \leq \alpha\}$ to define Y_α. Let $\{N_\zeta : \zeta < \kappa\}$ be a set of almost disjoint subsets of $\mathfrak{c}$ each of cardinality $\mathfrak{c}$. For any $\zeta < \kappa$ let $\{\zeta(\xi) : \xi < \mathfrak{c}\}$ be an increasing enumeration of N_ζ and define $f_\zeta : \mathbb{R} \to \mathbb{R}$ by $f_\zeta(x_\alpha) = y^\alpha_{\zeta(\alpha)}$. Let $\zeta_1 < \zeta_2 < \ldots < \zeta_n < \kappa$, P_β be a polynomial in n variables, g_γ be a Borel function and Z be any subset of $\mathbb{R}$ of cardinality $\mathfrak{c}$. There is $\xi < \mathfrak{c}$ such that $N_{\zeta_1}, N_{\zeta_2}, \ldots, N_{\zeta_n}$ are disjoint above ξ. Since Z is of cardinality $\mathfrak{c}$, there is $\alpha < \mathfrak{c}$ with $\alpha > \max\{\beta, \gamma, \xi\}$ and $x_\alpha \in Z$. Since α is greater than ξ, then $f_{\zeta_1}(x_\alpha), f_{\zeta_2}(x_\alpha), \ldots, f_{\zeta_n}(x_\alpha)$ are distinct points of Y_α. Since α is greater than β and γ, by construction $P_\beta(f_{\zeta_1}, f_{\zeta_2}, \ldots, f_{\zeta_n})$ differs from g_γ at the point $x_\alpha \in Z$. Therefore $P_\beta(f_{\zeta_1}, f_{\zeta_2}, \ldots, f_{\zeta_n})$ is a Sierpiński-Zygmund function.

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corollary

If one of the following set-theoretical assumption holds

- Martin's Axiom, or
- CH or,
- $\mathfrak{c}^+ = 2^{\mathfrak{c}}$,

then the set of Sierpiński-Zygmund functions is $2^{\mathfrak{c}}$-algebrable.

questions

1. Is it necessary to add any additional hypothesis to ZFC in order to obtain $2^{\mathfrak{c}}$-algebrability (or even $2^{\mathfrak{c}}$-lineability) of the set of Sierpiński-Zygmund functions?

2. Can one prove in ZFC that there is free subalgebra of $2^{\mathfrak{c}}$ generators in $\mathbb{R}^{\mathbb{R}}$?YES.

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theorem

$\mathbb{R}^{\mathbb{R}}$ contains a free linear algebra of $2^{\mathfrak{c}}$ generators.

Proof: Let

$$\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathbb{R}_*[x_1,\dots,x_n] \times n^\omega = \{(P_\alpha, p_\alpha) : \alpha < \mathfrak{c}\}.$$

For α choose a vector $\vec{x}_\alpha \in \mathbb{R}^n$ such that $P_\alpha(\vec{x}_\alpha) \neq 0$. $p_\alpha \in n^\omega$ admits a continuous extension $\bar{p}_\alpha : \beta\omega \to n$.Now to each ultrafilter $\mathcal{U} \in \beta\omega$ assign the function $f_{\mathcal{U}} : \mathfrak{c} \to \mathbb{R}$ defined by the formula

$$f_{\mathcal{U}}(\alpha) = \vec{x}_\alpha \circ \bar{p}_\alpha(\mathcal{U}).$$

We claim that the family $F = \{f_{\mathcal{U}}\}_{\mathcal{U}\in\beta\omega} \subset \mathbb{R}^{\mathfrak{c}}$ is algebraically independent.We need to check that $P(f_{\mathcal{U}_1},\dots,f_{\mathcal{U}_n}) \neq 0$ for any non-zero polynomial $P(x_1,\dots,x_n) \in \mathbb{R}_*[x_1,\dots,x_n]$ and any pairwise distinct ultrafilters $\mathcal{U}_1,\dots,\mathcal{U}_n \in \beta\omega$.

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$$\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathbb{R}_*[x_1, \ldots, x_n] \times n^\omega = \{(P_\alpha, p_\alpha) : \alpha < \mathfrak{c}\}.$$

For α choose a vector $\vec{x}_\alpha \in \mathbb{R}^n$ such that $P_\alpha(\vec{x}_\alpha) \neq 0$. $p_\alpha \in n^\omega$ admits a continuous extension $\bar{p}_\alpha : \beta\omega \to n$.Now to each ultrafilter $\mathcal{U} \in \beta\omega$ assign the function $f_\mathcal{U} : \mathfrak{c} \to \mathbb{R}$ defined by the formula

$$f_\mathcal{U}(\alpha) = \vec{x}_\alpha \circ \bar{p}_\alpha(\mathcal{U}).$$

We claim that the family $F = \{f_\mathcal{U}\}_{\mathcal{U}\in\beta\omega} \subset \mathbb{R}^{\mathfrak{c}}$ is algebraically independent.We need to check that $P(f_{\mathcal{U}_1}, \ldots, f_{\mathcal{U}_n}) \neq 0$ for any non-zero polynomial $P(x_1, \ldots, x_n) \in \mathbb{R}_*[x_1, \ldots, x_n]$ and any pairwise distinct ultrafilters $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \beta\omega$.

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