Supercompact cardinals and failures of GCH

Fusion and large cardinals

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Theorem

Theorem (Friedman, H., 2011)

(GCH) Assume $\kappa < \lambda$ are regular and κ is both λ-supercompact and λ^{++}-tall. Then there is a cofinality-preserving forcing P such that in V^P, κ is still λ-supercompact, GCH holds in $[\kappa, \lambda)$, but fails at λ.

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λ is regular can be a successor, even a successor of a singular cardinal: for more concreteness, you may assume $\lambda = \kappa^{+\omega+1}$.

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- Probably necessary for consistency of interesting combinatorial statements (such as PFA or MM).
- Lack of inner models – leaves forcing as the only technique. Related open questions: lower bound in consistency strength; forcing together *L*-like properties + and non *L*-like properties (such as definable wellorder plus failure of GCH).

Supercompact and tall cardinals

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Definition

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Notice that κ is measurable iff κ is κ-supercompact iff κ is κ-tall.

Supercompact and tall cardinals

Lemma

(GCH) Let $\kappa \leq \lambda$ be regular. Assume that κ is λ-supercompact and λ^{++}-tall. Then there exists $j : V \to M$ with critical point κ such that:

(i) ${}^{\lambda}M \subseteq M$;

(ii) $\lambda^{++} < j(\kappa) < \lambda^{+++}$;

(iii) $M = \{j(f)(j''\lambda, \alpha) \mid f : P_\kappa\lambda \times \kappa \to V \ \&\ \alpha < \lambda^{++}\}$.

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Notice that f's above have domains of size λ. In particular if E is in M a dense open set in $j(P)$ for some forcing $P \in V$, then E can be represented in V as a certain sequence $\langle D_i \mid i < \lambda \rangle$ of dense open sets in P.

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In order to preserve supercompactness, we look for a forcing P such that:

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This points to fusion-based forcings.

λ-Sacks forcing

Assume from now on that $\lambda = \lambda'^{+}$.

Definition

$S(\lambda)$, λ-Sacks forcing, a collection of "naturally defined" perfect trees in $2^{<\lambda}$ with $\leq$ equal to inclusion. $S(\lambda, \alpha)$ is the product with supports of size $\leq \lambda$.

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$$p \leq_{i,F_i} q \leftrightarrow p \leq q \;\&\; (\forall \beta \in F_i)\ {}^{i+1}2 \cap p(\beta) = {}^{i+1}2 \cap q(\beta).$$

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A decreasing sequence under $\leq_{i,F_i}$ of length λ, a **fusion sequence**, has the infimum – dubbed the **fusion limit**.

Basic fusion

To check $S(\lambda, \alpha)$ preserves λ^+, we first fix a dimond sequence:

Definition

Let us fix a $\diamondsuit_\lambda$ sequence

$$\langle S_i \,|\, i < \lambda \,\&\, S_i \subseteq i \times i \rangle.$$

For every $A \subseteq \lambda \times \lambda$, the set $\{i < \lambda \,|\, S_i = A \cap (i \times i)\}$ is stationary.

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Note that $\diamondsuit_\lambda$ is implied by GCH at λ'.

Basic reduction lemma

Lemma (Basic reduction lemma)

Assume p is in $S(\lambda, \alpha)$ *and* $\langle D_i \mid i < \lambda \rangle$ *is a sequence of dense open sets. Then there exists a condition* $q \leq p$, $q = fusionlim(p_i)_{i<\lambda}$, *such that for any* $i < \lambda$ *and* **any** $t \leq q$ *there* **exists** $j > i$ *such that the* **restrictions of** q **and** t **to** S_j *are defined and both are in* D_i.

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Compare with the case when the cardinal is inaccessible:

Lemma (κ inaccessible, or ω)

Assume p is in $S(\kappa, \alpha)$ and $\langle D_i \mid i < \kappa \rangle$ is a sequence of dense open sets. Then there exists a condition $q \leq p$, $q = \mathit{fusionlimit}(p_i)_{i<\kappa}$, such that if r is **any** *thinning of q to stems of height i (on a certain $< \kappa$ big subset of support of q), then r is in D_i.*

Coherent sequences

Definition

Fix p and $F = \bigcup F_n \subseteq \text{support}(p)$, with $|F_n| < \lambda$ for every $n < \omega$. Let $i < \lambda$ have cof ω and let $\langle i_n \mid n < \omega \rangle$ be cofinal in i. We say that a sequence $\langle S_{i_n} \mid n < \omega \rangle$ is **coherent** with respect to p and F if the family $\{S_{i_n}(\delta) \restriction i_{n-1} \mid \delta(n) < n < \omega\}$ determines an element of ${}^{i}2$ for each δ in F. (Where $\delta(n)$ is the least n such that δ is in F_n.)

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Notice that if $\text{cf}(\lambda') > \omega$, then the number of all sequences $\langle i_n \,|\, n < \omega \rangle$ cofinal in i is at most λ', and so is the number of resulting coherent sequences. (If $\text{cf}(\lambda') = \omega$, a little more needs to be done.)

Rich reduction lemma

Lemma (Rich reduction lemma)

Assume p is in $S(\lambda,\alpha)$ and $\langle D_i \mid i < \lambda\rangle$ is a sequence of dense open sets. Then there exists a condition $q \le p$, $q = fusionlim(p_i)_{i<\lambda}$, which is a basic reduction and moreover: for every i, if $j > i$ has cofinality ω and $j+1$ was a non-trivial stage of construction, then for every coherent sequence $\langle S_{j_n} \mid i < n\rangle$, q restricted to the nodes determined by this sequence (if this restriction makes sense) lies in D_i.

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See blackboard for a "hand-waving proof" that this is enough to prove the theorem.

An open question

It was crucial for the proof that the **length of the fusion** in $S(\lambda, \lambda^{++})$ was equal to the **support** of $j : V \to M$ (the support of j equals the size of the domains of the relevant f's describing M). For instance, this technique does not work for $S(\kappa, \lambda^{++})$ – too short a fusion, too few clubs in κ.

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Question. Is there a κ-closed cofinality-preserving forcing P which adds new subsets of κ, but supports a "genuine" fusion of length μ for cardinals $\mu \in [\kappa, \lambda]$? One can use that κ is λ-supercompact.