Kuratowski operations re-visit

by

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Section Set Theory & Topology
My research (doctoral - announced to me by the co-author of this talk) project is:

Let $f : X \to Y$ be a skeletal map - a notion (its name) was introduced by J. Mioduszewski and L. Rudolf (1969). Describe monoids generated by words

"$\sigma_i \circ f^{-1} \circ \sigma_j \circ f \circ \sigma_k$" or "$\sigma_i \circ f \circ \sigma_j \circ f^{-1} \circ \sigma_k$",

where $\sigma_n$ are Kuratowski operations.

A skeletal function is continuous and the closure of the image $f[U]$ is regularly closed, for any open set $U \subseteq X$ - or the preimage of a closed and nowhere dense set is nowhere dense.
So, one can use the following cancellations, where "−" denotes operations of closure (in $X$ or $Y$) and "$c$" operations of complement (with respect to $X$ or $Y$):

### Some possible cancellations

- $f^{-1}(V^-) = f^{-1}(V^-)^- \text{ and } f^{-1}(V^-c) = f^{-1}(V^-c)c^-c$ - whenever $f$ is continuous;
- $f[V^-] = f[V^-]^-$ - whenever $f$ is a closed map;
- $f[V^-c] = f[V^-c]c^-c$ - whenever $f$ is an open map;
- $f^{-1}(V^-c)^-c^-c = f^{-1}(V^-c)^c^-c$ and $f[V^-c]^- = f[V^-c]^c^-c$ - whenever $f$ is skeletal and continuous;
- $f^{-1}(V^-c) = f^{-1}(V^-c)^c^-c$ - whenever $f$ is d-open and continuous.
Kuratowski operations are compositions of "\(-\)" and "\(c\)". K. Kuratowski presented examination of these operations in his dissertation and in the article "Sur l'opération "A de l'Analysis Situs", Fund. Math. (1922). There are 14 such operations:

\[
\begin{align*}
\sigma_0(A) &= A \\
\sigma_1(A) &= A^c \text{ (complement)} \\
\sigma_2(A) &= A^- \text{ (closure)} \\
\sigma_3(A) &= A^{c^-} \\
\sigma_4(A) &= A^{-c} \\
\sigma_5(A) &= A^{c-c} \text{ (interior)} \\
\sigma_6(A) &= A^{-c^-} \\
\sigma_7(A) &= A^{c-c^-} \\
\sigma_8(A) &= A^{-c-c} \\
\sigma_9(A) &= A^{c-c-c} \\
\sigma_{10}(A) &= A^{-c-c^-} \\
\sigma_{11}(A) &= A^{c-c-c^-} \\
\sigma_{12}(A) &= A^{-c-c-c} \\
\sigma_{13}(A) &= A^{c-c-c-c} \\
A^{-c-} &= A^{-c-c-c-} \\
A^{c-c-} &= A^{c-c-c-c-} 
\end{align*}
\]
The Kuratowski operations form the monoid $M$ with the following table of compositions:

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where in the $i$-th row (marked by $\sigma_i$) and the $k$-th column is $\sigma_i \circ \sigma_k$. 
Kuratowski operations have been concern of many authors, students have to consider it as classical exercises of topology, etc. B. J. Gardner and M. Jackson wrote a survey article “The Kuratowski closure-complement theorem” in New Zealand J. Math. 38 (2008), pp. 9 – 44. The last (published) article is ”Variations on Kuratowski’s 14-set theorem” by D. Sherman in Amer. Math. Monthly 117(2) (2010), pp. 113 - 123.

Unfortunately, we could not find the list of all semigroups contained in the monoid $M$ !?!.  

At first sight, to prepare a suitable list seems not more complicated than to solve a sudoku puzzle. But some questions about sudoku puzzles have been unsolved. For example, A. M. Herzberg and M. R. Murty wrote: It is unknown at present if a puzzle with 16 specified entries exists that yields a unique solution; in Notices of the AMS 54(6) (2007), pp. 708.
Since this moment, I would like to describe the list of all semigroups contained in $\mathbb{M}$, using a pencil-and-paper techniques only.

Note that, the operation $\sigma_1(A) = A^c$ is a decreasing (i.e. $A \subseteq B \Rightarrow A^c \supseteq B^c$) involution (i.e. $A^{cc} = A$) and $\sigma_2(A) = A^-$ is a increasing (i.e. $A \subseteq B \Rightarrow A^- \subseteq B^-$) idempotent (i.e. $A^{--} = A^-$) function such that $A \subseteq A^-$. These properties give cancellations:

- $\sigma_6(A) = A^{--c} = A^{c--c--} = (\sigma_{12}(A))^-;$
- $\sigma_7(A) = A^{c-c} = A^{c-c-c-c-} = (\sigma_{13}(A))^-.$
Browse through semigroup with one generator we obtain 12 groups:

- $G_0 = \langle \sigma_0 \rangle, G_2 = \langle \sigma_2 \rangle, G_5 = \langle \sigma_5 \rangle, G_7 = \langle \sigma_7 \rangle, G_8 = \langle \sigma_8 \rangle, G_{10} = \langle \sigma_{10} \rangle$ and $G_3 = \langle \sigma_{13} \rangle$ - groups with one element;
- $G_1 = \langle \sigma_1 \rangle = \{\sigma_0, \sigma_1\}, G_6 = \langle \sigma_6 \rangle = \{\sigma_6, \sigma_{10}\}, G_{11} = \langle \sigma_{11} \rangle = \{\sigma_7, \sigma_{11}\}, G_4 = \langle \sigma_{12} \rangle = \{\sigma_8, \sigma_{12}\}$ and $G_9 = \langle \sigma_9 \rangle = \{\sigma_9, \sigma_{13}\}$ - groups with two elements,

and 2 semigroups with three elements:

- $S_3 = \langle \sigma_3 \rangle = \{\sigma_3, \sigma_7, \sigma_{11}\};$
- $S_4 = \langle \sigma_4 \rangle = \{\sigma_4, \sigma_8, \sigma_{12}\}.$
Each semigroup containing $\sigma_1$ (or $\sigma_0$) is a monoid, since $\sigma_1 \circ \sigma_1 = \sigma_0$.

**Theorem**

There are three monoids containing $\sigma_1$:

- $G_1 = < \sigma_1 > = \{\sigma_0, \sigma_1\}$;
- $M = < \sigma_1, \sigma_i > = \{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{13}\}$, where $i \in \{2, 3, 4, 5\}$;
- $M_1 = < \sigma_1, \sigma_j > = \{\sigma_0, \sigma_1, \sigma_6, \sigma_7 \ldots, \sigma_{13}\}$, where $j \in \{6, 7, \ldots, 13\}$.

Delete rows and columns marked by $\sigma_1$ in the table of compositions. Then, check that the operation $\sigma_3$ is in the row or the column marked by $\sigma_3$ only; the operation $\sigma_4$ is in the row or the column marked by $\sigma_4$ only.
Theorem

The semigroup

\( S_2 = <\sigma_3, \sigma_4 >= \{\sigma_2, \sigma_3, \ldots, \sigma_{13}\} \),

has the unique minimal set of generators \( \{\sigma_3, \sigma_4\} \).

Automorphism \( A \) swaps closure for interior, etc.

The permutation

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
0 & 1 & 5 & 4 & 3 & 2 & 9 & 8 & 7 & 6 & 13 & 12 & 11 & 10
\end{pmatrix}
\]

determines the automorphism \( A : M \rightarrow M \), where \( A[S_2] = S_2 \), whenever indexes of \( \sigma_i \) are according to this permutation.
The semigroup $S_6 = \{\sigma_6, \sigma_7, \ldots, \sigma_{13}\}$

Each operation $\sigma_k$, where $6 \leq k$, has to be a composition (of $\sigma_1$ and $\sigma_2$) which consists of 2 or more times the closure operation $\sigma_2$. So, $\{\sigma_6, \sigma_7, \ldots, \sigma_{13}\} = S_6$ is a semigroup with the table of compositions:

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Minimal sets of generators for $S_6$

The semigroup $S_6$ has the following minimal sets of generators:

- **(2-elements)** $\langle \sigma_6, \sigma_9 \rangle$, $\langle \sigma_6, \sigma_{13} \rangle$, $\langle \sigma_7, \sigma_{12} \rangle$, $\langle \sigma_8, \sigma_{11} \rangle$, $\langle \sigma_9, \sigma_{10} \rangle$ and $\langle \sigma_{11}, \sigma_{12} \rangle$;
- **(3-elements)** $\langle \sigma_6, \sigma_7, \sigma_8 \rangle$, $\langle \sigma_7, \sigma_8, \sigma_9 \rangle$, $\langle \sigma_{10}, \sigma_{11}, \sigma_{13} \rangle$ and $\langle \sigma_{10}, \sigma_{12}, \sigma_{13} \rangle$.

The semigroup $S_6$ contains four semigroups, each one contains two elements, which are idempotents:

- $I_1 = \langle \sigma_7, \sigma_{10} \rangle = \{\sigma_7, \sigma_{10}\}$;
- $I_2 = \langle \sigma_7, \sigma_{13} \rangle = \{\sigma_7, \sigma_{13}\}$;
- $I_3 = \langle \sigma_8, \sigma_{10} \rangle = \{\sigma_8, \sigma_{10}\}$;
- $I_4 = \langle \sigma_8, \sigma_{13} \rangle = \{\sigma_8, \sigma_{13}\}$.

$S_6$ contains also five semigroups, each one contains four elements and has 2-elements (minimal) sets of generators only.
Let us list these semigroup.

Denote by $S_7$ the semigroup with the table of compositions

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The semigroup $S_7 = \{\sigma_6, \sigma_7, \sigma_{10}, \sigma_{11}\}$ has three sets of (minimal) generators: $<\sigma_6, \sigma_7>$, $<\sigma_6, \sigma_{11}>$ and $<\sigma_{10}, \sigma_{11}>$. The image of $S_7$ is denoted by $S_5 = \{\sigma_8, \sigma_9, \sigma_{12}, \sigma_{13}\} = A[S_7]$. The semigroup $S_5$ has three sets of (minimal) generators: $<\sigma_8, \sigma_9>$, $<\sigma_9, \sigma_{12}>$ and $<\sigma_{12}, \sigma_{13}>$. 
The semigroup with the table of compositions

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is denoted by $S_8 = \{\sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$. This semigroup has three sets of (minimal) generators: $<\sigma_6, \sigma_8>$, $<\sigma_6, \sigma_{12}>$ and $<\sigma_{10}, \sigma_{12}>$. The image of $S_8$ is denoted by $S_9 = \{\sigma_7, \sigma_9, \sigma_{11}, \sigma_{13}\} = A[S_8]$. The semigroup $S_9$ has three sets of (minimal) generators: $<\sigma_7, \sigma_9>$, $<\sigma_9, \sigma_{11}>$ and $<\sigma_{11}, \sigma_{13}>$. 
The semigroup $S_1 = \{\sigma_7, \sigma_8, \sigma_{10}, \sigma_{13}\}$ has two sets of (minimal) generators: $<\sigma_7, \sigma_8>$ and $<\sigma_{10}, \sigma_{13}>$. The table of composition for $S_1$ is presented below.

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Thus the semigroup $S_6$ contains 8 groups: $G_3, G_4, G_6, G_7, G_8, G_9, G_{10}$ and $G_{11}$; and 10 semigroups: $I_1, I_2, I_3, I_4, S_1, S_5, S_6, S_7, S_8$ and $S_9$. In amount, $S_6$ contains eighteen semigroups.
Remaining groups with two generators in $S_2$

Monoids

- $< \sigma_2, \sigma_6 > = \{ \sigma_2, \sigma_6, \sigma_{10} \} = M_2 = A[M_5]$;
- $< \sigma_2, \sigma_{10} > = \{ \sigma_2, \sigma_{10} \} = M_3 = A[M_4]$;
- $< \sigma_5, \sigma_{13} > = \{ \sigma_5, \sigma_{13} \} = M_4 = A[M_3]$;
- $< \sigma_5, \sigma_9 > = \{ \sigma_5, \sigma_9, \sigma_{13} \} = M_5 = A[M_2]$.

Non-monoids

- $< \sigma_2, \sigma_7 > = \{ \sigma_2, \sigma_7, \sigma_{10} \} = II_5 = A[II_8]$;
- $< \sigma_2, \sigma_8 > = \{ \sigma_2, \sigma_8, \sigma_{10} \} = II_6 = A[II_7]$;
- $< \sigma_5, \sigma_7 > = \{ \sigma_5, \sigma_7, \sigma_{13} \} = II_7 = A[II_6]$;
- $< \sigma_5, \sigma_8 > = \{ \sigma_5, \sigma_8, \sigma_{13} \} = II_8 = A[II_5]$. 
\[ < \sigma_2, \sigma_{11} > = < \sigma_2, \sigma_6, \sigma_7 > = \{ \sigma_2, \sigma_6, \sigma_7, \sigma_{10}, \sigma_{11} \} = \{ \sigma_2 \} \cup S_7. \]

\[ < \sigma_2, \sigma_{12} > = < \sigma_2, \sigma_6, \sigma_8 > = \{ \sigma_2, \sigma_6, \sigma_8, \sigma_{10}, \sigma_{12} \} = \{ \sigma_2 \} \cup S_8. \]

\[ < \sigma_2, \sigma_{13} > = < \sigma_2, \sigma_7, \sigma_8 > = \{ \sigma_2, \sigma_7, \sigma_8, \sigma_{10}, \sigma_{13} \} = \{ \sigma_2 \} \cup S_1. \]

\[ < \sigma_2, \sigma_9 > = \{ \sigma_2 \} \cup S_6. \]

Sets of generators: \( < \sigma_2, \sigma_6, \sigma_{13} >, < \sigma_2, \sigma_7, \sigma_{12} >, < \sigma_2, \sigma_8, \sigma_{11} >, < \sigma_2, \sigma_{11}, \sigma_{12} >, < \sigma_2, \sigma_{11}, \sigma_{13} > \) and \( < \sigma_2, \sigma_{12}, \sigma_{13} > \) are minimal in \( \{ \sigma_2 \} \cup S_6. \)
\[ A[<\sigma_2, \sigma_{11}>] = <\sigma_5, \sigma_{12}> = <\sigma_5, \sigma_9, \sigma_8> = \{\sigma_5, \sigma_9, \sigma_8, \sigma_{13}, \sigma_{12}\} = \{\sigma_5\} \cup S_5; \]

\[ A[<\sigma_2, \sigma_{12}>] = <\sigma_5, \sigma_{11}> = <\sigma_5, \sigma_9, \sigma_7> = \{\sigma_5, \sigma_9, \sigma_7, \sigma_{13}, \sigma_{11}\} = \{\sigma_5\} \cup S_9; \]

\[ A[<\sigma_2, \sigma_{13}>] = <\sigma_5, \sigma_{10}> = <\sigma_5, \sigma_8, \sigma_7> = \{\sigma_5, \sigma_8, \sigma_7, \sigma_{13}, \sigma_{10}\} = \{\sigma_5\} \cup S_1. \]

\[ A[<\sigma_2, \sigma_9>] = <\sigma_5, \sigma_6> = \{\sigma_5\} \cup S_6 = <\sigma_5, \sigma_9, \sigma_{10}> = <\sigma_5, \sigma_8, \sigma_{11}> = <\sigma_5, \sigma_7, \sigma_{12}> = <\sigma_5, \sigma_{12}, \sigma_{11}> = <\sigma_5, \sigma_{12}, \sigma_{10}> = <\sigma_5, \sigma_{11}, \sigma_{10}>. \]
• $< \sigma_3, \sigma_6 > = < \sigma_3, \sigma_{10} > = \{ \sigma_3, \sigma_6, \sigma_7 \sigma_{10}, \sigma_{11} \} = \{ \sigma_3 \} \cup S_7$;
• $< \sigma_3, \sigma_9 > = < \sigma_3, \sigma_{13} > = \{ \sigma_3, \sigma_7, \sigma_9 \sigma_{11}, \sigma_{13} \} = \{ \sigma_3 \} \cup S_9$;
• $< \sigma_3, \sigma_8 > = < \sigma_3, \sigma_{12} > = \{ \sigma_3 \} \cup S_6$.

\[ A[< \sigma_3, \sigma_6 >] = < \sigma_4, \sigma_9 > = < \sigma_4, \sigma_{13} > = \{ \sigma_4, \sigma_8, \sigma_9 \sigma_{12}, \sigma_{13} \} = \{ \sigma_4 \} \cup S_5; \]

\[ A[< \sigma_3, \sigma_9 >] = < \sigma_4, \sigma_6 > = < \sigma_4, \sigma_{10} > = \{ \sigma_4, \sigma_6, \sigma_8 \sigma_{10}, \sigma_{12} \} = \{ \sigma_4 \} \cup S_8; \]

\[ A[< \sigma_3, \sigma_8 >] = < \sigma_4, \sigma_7 > = < \sigma_4, \sigma_{11} > = \{ \sigma_4 \} \cup S_6. \]

Additionally,

• $\{ \sigma_3 \} \cup S_6 = < \sigma_3, \sigma_6, \sigma_9 > = < \sigma_3, \sigma_6, \sigma_{13} > = < \sigma_3, \sigma_9, \sigma_{10} > = < \sigma_3, \sigma_{10}, \sigma_{13} >$;

• $\{ \sigma_4 \} \cup S_6 = < \sigma_4, \sigma_6, \sigma_9 > = < \sigma_4, \sigma_9, \sigma_{10} > = < \sigma_4, \sigma_6, \sigma_{13} > = < \sigma_4, \sigma_{10}, \sigma_{13} >$. 

Kuratowski operations re-visit
\[< \sigma_2, \sigma_3 > = \{ \sigma_2, \sigma_3, \sigma_6, \sigma_7, \sigma_{10} \sigma_{11} \} = \{ \sigma_2, \sigma_3 \} \cup S_7.\]

\[< \sigma_2, \sigma_3, \sigma_8 > = < \sigma_2, \sigma_3, \sigma_9 > = < \sigma_2, \sigma_3, \sigma_{12} > = < \sigma_2, \sigma_3, \sigma_{13} > = \{ \sigma_2, \sigma_3 \} \cup S_6.\]

\[A[< \sigma_3, \sigma_2 >] = < \sigma_4, \sigma_5 > = \{ \sigma_4, \sigma_5, \sigma_8, \sigma_9, \sigma_{12} \sigma_{13} \} = \{ \sigma_4, \sigma_5 \} \cup S_5.\]

\[A[< \sigma_3, \sigma_2, \sigma_8 >] = < \sigma_4, \sigma_5, \sigma_7 > = < \sigma_4, \sigma_5, \sigma_6 > = < \sigma_4, \sigma_5, \sigma_{11} > = < \sigma_4, \sigma_5, \sigma_{10} > = \{ \sigma_4, \sigma_5 \} \cup S_6.\]

\[< \sigma_2, \sigma_4 > = \{ \sigma_2, \sigma_4, \sigma_6, \sigma_8, \sigma_{10} \sigma_{12} \} = \{ \sigma_2, \sigma_4 \} \cup S_8;\]

\[< \sigma_2, \sigma_4, \sigma_7 > = < \sigma_2, \sigma_4, \sigma_9 > = < \sigma_2, \sigma_4, \sigma_{11} > = < \sigma_2, \sigma_4, \sigma_{13} > = \{ \sigma_2, \sigma_4 \} \cup S_6;\]
Thus, the algorithm which enumerates all semigroups (without \(\sigma_0\)) is ready. One can check that it enumerates four (4) monoids with \(\sigma_0\) and fifty seven (57) semigroups without \(\sigma_0\). Thus, the monoid \(\mathbb{M}\) contains \(118 = 4 + 57 \cdot 2\) semigroups.
Exercises easy!? 

Let $X$ be an extremally disconnected space, then $\sigma_6 = \sigma_{12}$, $\sigma_7 = \sigma_{13}$, $\sigma_8 = \sigma_{10}$ and $\sigma_9 = \sigma_{11}$. How many errors does contain the following table of compositions of Kuratowski operations?

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If you believe that it does not have errors, then find (describe) and count all semigroups in monoids $\mathbb{M}$ for $X$. There are 70 semigroups!

Have I a several years to crack (using pencil and paper only) the problem: Describe monoids generated by words

"$\sigma_i \circ f^{-1} \circ \sigma_j \circ f \circ \sigma_k$" or "$\sigma_i \circ f \circ \sigma_j \circ f^{-1} \circ \sigma_k$",

where $\sigma_n$ are Kuratowski operations?