Kuratowski operations re-visit

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My research (doctoral - announced to me by the co-author of this talk) project is:

Let $f : X \to Y$ be a skeletal map - a notion (its name) was introduced by J. Mioduszewski and L. Rudolf (1969). Describe monoids generated by words

$$"\sigma_i \circ f^{-1} \circ \sigma_j \circ f \circ \sigma_k" \text{ or } "\sigma_i \circ f \circ \sigma_j \circ f^{-1} \circ \sigma_k",$$

where σ_n are Kuratowski operations.

A *skeletal function* is continuous and the closure of the image $f[U]$ is regularly closed, for any open set $U \subseteq X$ - or the preimage of a closed and nowhere dense set is nowhere dense.

So, one can use the following cancellations, where $"^-"$ denotes operations of closure (in X or Y) and $"^c"$ operations of complement (with respect to X or Y):

Some possible cancellations

- $f^{-1}(V^{-}) = f^{-1}(V^{-})^{-}$ and $f^{-1}(V^{-c}) = f^{-1}(V^{-c})^{c-c}$ - whenever f is continuous;
- $f[V^{-}] = f[V^{-}]^{-}$ - whenever f is a closed map;
- $f[V^{-c}] = f[V^{-c}]^{c-c}$ - whenever f is an open map;
- $f^{-1}(V^{-c-})^{c-c} = f^{-1}(V^{-c})^{-c-c}$ and $f[V^{-c}]^{-} = f[V^{-c}]^{-c-c-}$ - whenever f is skeletal and continuous;
- $f^{-1}(V^{-c}) = f^{-1}(V^{-c})^{-c-c}$ - whenever f is d-open and continuous.

Kuratowski operations are compositions of "$-$" and "c". K. Kuratowski presented examination of these operations in his dissertation and in the article "*Sur l'opération $\overline{A}$ de l'Analysis Situs*", Fund. Math. (1922). There are 14 such operations:

$\sigma_0(A)$	$=$	A	$\sigma_1(A)$	$=$	A^c (complement)
$\sigma_2(A)$	$=$	A^- (closure)	$\sigma_3(A)$	$=$	A^{c-}
$\sigma_4(A)$	$=$	A^{-c}	$\sigma_5(A)$	$=$	A^{c-c} (interior)
$\sigma_6(A)$	$=$	A^{-c-}	$\sigma_7(A)$	$=$	A^{c-c-}
$\sigma_8(A)$	$=$	A^{-c-c}	$\sigma_9(A)$	$=$	A^{c-c-c}
$\sigma_{10}(A)$	$=$	A^{-c-c-}	$\sigma_{11}(A)$	$=$	A^{c-c-c-}
$\sigma_{12}(A)$	$=$	A^{-c-c-c}	$\sigma_{13}(A)$	$=$	$A^{c-c-c-c}$
A^{-c-}	$=$	$A^{-c-c-c-}$	A^{c-c-}	$=$	$A^{c-c-c-c-}$

The Kuratowski operations form the monoid $\mathbb{M}$ with the following table of compositions:

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
σ_1	σ_0	σ_4	σ_5	σ_2	σ_3	σ_8	σ_9	σ_6	σ_7	σ_{12}	σ_{13}	σ_{10}	σ_{11}
σ_2	σ_3	σ_2	σ_3	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7
σ_3	σ_2	σ_6	σ_7	σ_2	σ_3	σ_{10}	σ_{11}	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}
σ_4	σ_5	σ_4	σ_5	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9
σ_5	σ_4	σ_8	σ_9	σ_4	σ_5	σ_{12}	σ_{13}	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}
σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}
σ_7	σ_6	σ_{10}	σ_{11}	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7
σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}
σ_9	σ_8	σ_{12}	σ_{13}	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9
σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7
σ_{11}	σ_{10}	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}
σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9
σ_{13}	σ_{12}	σ_8	σ_9	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}

where in the i-th row (marked by σ_i) and the k-th column is $\sigma_i \circ \sigma_k$.

Kuratowski operations have been concern of many authors, students have to consider it as classical exercises of topology, etc. B. J. Gardner and M. Jackson wrote a survey article "*The Kuratowski closure-complement theorem*" in New Zealand J. Math. 38 (2008), pp. 9 – 44. The last (published) article is "*Variations on Kuratowski's 14-set theorem*" by D. Sherman in Amer. Math. Monthly 117(2) (2010), pp. 113 - 123.

Unfortunately, we could not find the list of all semigroups contained in the monoid $\mathbb{M}$!?!.

At first sight, to prepare a suitable list seems not more complicated than to solve a sudoku puzzle. But some questions about sudoku puzzles have been unsolved. For example, A. M. Herzberg and M. R. Murty wrote: *It is unknown at present if a puzzle with 16 specified entries exists that yields a unique solution;* in Notices of the AMS 54(6) (2007), pp. 708.

Since this moment, I would like to describe the list of all semigroups contained in $\mathbb{M}$, using a pencil-and-paper techniques only.

Note that, the operation $\sigma_1(A) = A^c$ is a decreasing (i.e. $A \subseteq B \Rightarrow A^c \supseteq B^c$) involution (i.e. $A^{cc} = A$) and $\sigma_2(A) = A^-$ is a increasing (i.e. $A \subseteq B \Rightarrow A^- \subseteq B^-$) idempotent (i.e. $A^{--} = A^-$) function such that $A \subseteq A^-$. These properties give cancellations:

- $\sigma_6(A) = A^{-c-} = A^{-c-c-c-} = (\sigma_{12}(A))^-$;
- $\sigma_7(A) = A^{c-c-} = A^{c-c-c-c-} = (\sigma_{13}(A))^-$.

Browse through semigroup with one generator we obtain 12 groups:

- $\mathbb{G}_0 =< \sigma_0 >$, $\mathbb{G}_2 =< \sigma_2 >$, $\mathbb{G}_5 =< \sigma_5 >$, $\mathbb{G}_7 =< \sigma_7 >$, $\mathbb{G}_8 =< \sigma_8 >$, $\mathbb{G}_{10} =< \sigma_{10} >$ and $\mathbb{G}_3 =< \sigma_{13} >$ - groups with one element;
- $\mathbb{G}_1 =< \sigma_1 >= \{\sigma_0, \sigma_1\}$, $\mathbb{G}_6 =< \sigma_6 >= \{\sigma_6, \sigma_{10}\}$, $\mathbb{G}_{11} =< \sigma_{11} >= \{\sigma_7, \sigma_{11}\}$, $\mathbb{G}_4 =< \sigma_{12} >= \{\sigma_8, \sigma_{12}\}$ and $\mathbb{G}_9 =< \sigma_9 >= \{\sigma_9, \sigma_{13}\}$ - groups with two elements,

and 2 semigroups with three elements:

- $\mathbb{S}_3 =< \sigma_3 >= \{\sigma_3, \sigma_7, \sigma_{11}\}$;
- $\mathbb{S}_4 =< \sigma_4 >= \{\sigma_4, \sigma_8, \sigma_{12}\}$.

Each semigroup containing σ_1 (or σ_0) is a monoid, since $\sigma_1 \circ \sigma_1 = \sigma_0$.

Theorem

There are three monoids containing σ_1:

- $\mathbb{G}_1 =< \sigma_1 >= \{\sigma_0, \sigma_1\}$;
- $\mathbb{M} =< \sigma_1, \sigma_i >= \{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{13}\}$, where $i \in \{2, 3, 4, 5\}$;
- $\mathbb{M}_1 =< \sigma_1, \sigma_j >= \{\sigma_0, \sigma_1, \sigma_6, \sigma_7 \ldots, \sigma_{13}\}$, where $j \in \{6, 7, \ldots, 13\}$.

Delete rows and columns marked by σ_1 in the table of compositions. Than, check that the operation σ_3 is in the row or the column marked by σ_3 only; the operation σ_4 is in the row or the column marked by σ_4 only.

Theorem

The semigroup

$$\mathbb{S}_2 =< \sigma_3, \sigma_4 >= \{\sigma_2, \sigma_3, \dots, \sigma_{13}\},$$

has the unique minimal set of generators $\{\sigma_3, \sigma_4\}$.

Automorphism $\mathbb{A}$ swaps closure for interior, etc.

The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 0 & 1 & 5 & 4 & 3 & 2 & 9 & 8 & 7 & 6 & 13 & 12 & 11 & 10 \end{pmatrix}$$

determines the automorphism $\mathbb{A} : \mathbb{M} \to \mathbb{M}$, where $\mathbb{A}[\mathbb{S}_2] = \mathbb{S}_2$, whenever indexes of σ_i are according to this permutation.

The semigroup $\mathbb{S}_6 = \{\sigma_6, \sigma_7, \ldots, \sigma_{13}\}$

Each operation σ_k, where $6 \leq k$, has to be a composition (of σ_1 and σ_2) which consists of 2 or more times the closure operation σ_2. So, $\{\sigma_6, \sigma_7, \ldots, \sigma_{13}\} = \mathbb{S}_6$ is a semigroup with the table of compositions:

	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
σ_6	σ_{10}	σ_{11}	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}
σ_7	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7
σ_8	σ_{12}	σ_{13}	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}
σ_9	σ_8	σ_9	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9
σ_{10}	σ_6	σ_7	σ_{10}	σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7
σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7	σ_6	σ_7	σ_{10}	σ_{11}
σ_{12}	σ_8	σ_9	σ_{12}	σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9
σ_{13}	σ_{12}	σ_{13}	σ_8	σ_9	σ_8	σ_9	σ_{12}	σ_{13}

Minimal sets of generators for $\mathbb{S}_6$

The semigroup $\mathbb{S}_6$ has the following minimal sets of generators:

- (2-elements) $<\sigma_6, \sigma_9>$, $<\sigma_6, \sigma_{13}>$, $<\sigma_7, \sigma_{12}>$, $<\sigma_8, \sigma_{11}>$, $<\sigma_9, \sigma_{10}>$ and $<\sigma_{11}, \sigma_{12}>$;
- (3-elements) $<\sigma_6, \sigma_7, \sigma_8>$, $<\sigma_7, \sigma_8, \sigma_9>$, $<\sigma_{10}, \sigma_{11}, \sigma_{13}>$ and $<\sigma_{10}, \sigma_{12}, \sigma_{13}>$.

The semigroup $\mathbb{S}_6$ contains four semigroups, each one contains two elements, which are idempotents:

- $\mathbb{I}_1 =< \sigma_7, \sigma_{10} >= \{\sigma_7, \sigma_{10}\}$;
- $\mathbb{I}_2 =< \sigma_7, \sigma_{13} >= \{\sigma_7, \sigma_{13}\}$;
- $\mathbb{I}_3 =< \sigma_8, \sigma_{10} >= \{\sigma_8, \sigma_{10}\}$;
- $\mathbb{I}_4 =< \sigma_8, \sigma_{13} >= \{\sigma_8, \sigma_{13}\}$.

$\mathbb{S}_6$ contains also five semigroups, each one contains four elements and has 2-elements (minimal) sets of generators only.

Let us list these semigroup.

Denote by $\mathbb{S}_7$ the semigroup with the table of compositions

	σ_6	σ_7	σ_{10}	σ_{11}
σ_6	σ_{10}	σ_{11}	σ_6	σ_7
σ_7	σ_6	σ_7	σ_{10}	σ_{11}
σ_{10}	σ_6	σ_7	σ_{10}	σ_{11}
σ_{11}	σ_{10}	σ_{11}	σ_6	σ_7

The semigroup $\mathbb{S}_7 = \{\sigma_6, \sigma_7, \sigma_{10}, \sigma_{11}\}$ has three sets of (minimal) generators: $< \sigma_6, \sigma_7 >, < \sigma_6, \sigma_{11} >$ and $< \sigma_{10}, \sigma_{11} >$. The image of $\mathbb{S}_7$ is denoted by $\mathbb{S}_5 = \{\sigma_8, \sigma_9, \sigma_{12}, \sigma_{13}\} = \mathbb{A}[\mathbb{S}_7]$. The semigroup $\mathbb{S}_5$ has three sets of (minimal) generators: $< \sigma_8, \sigma_9 >, < \sigma_9, \sigma_{12} >$ and $< \sigma_{12}, \sigma_{13} >$.

The semigroup with the table of compositions

	σ_6	σ_8	σ_{10}	σ_{12}
σ_6	σ_{10}	σ_6	σ_6	σ_{10}
σ_8	σ_{12}	σ_8	σ_8	σ_{12}
σ_{10}	σ_6	σ_{10}	σ_{10}	σ_6
σ_{12}	σ_8	σ_{12}	σ_{12}	σ_8

is denoted by $\mathbb{S}_8 = \{\sigma_6, \sigma_8, \sigma_{10}, \sigma_{12}\}$. This semigroup has three sets of (minimal) generators: $< \sigma_6, \sigma_8 >, < \sigma_6, \sigma_{12} >$ and $< \sigma_{10}, \sigma_{12} >$. The image of $\mathbb{S}_8$ is denoted by $\mathbb{S}_9 = \{\sigma_7, \sigma_9, \sigma_{11}, \sigma_{13}\} = \mathbb{A}[\mathbb{S}_8]$. The semigroup $\mathbb{S}_9$ has three sets of (minimal) generators: $< \sigma_7, \sigma_9 >, < \sigma_9, \sigma_{11} >$ and $< \sigma_{11}, \sigma_{13} >$.

The semigroup $\mathbb{S}_1 = \{\sigma_7, \sigma_8, \sigma_{10}, \sigma_{13}\}$ has two sets of (minimal) generators: $< \sigma_7, \sigma_8 >$ and $< \sigma_{10}, \sigma_{13} >$. The table of composition for $\mathbb{S}_1$ is presented below.

	σ_7	σ_8	σ_{10}	σ_{13}
σ_7	σ_7	σ_{10}	σ_{10}	σ_7
σ_8	σ_{13}	σ_8	σ_8	σ_{13}
σ_{10}	σ_7	σ_{10}	σ_{10}	σ_7
σ_{13}	σ_{13}	σ_8	σ_8	σ_{13}

Thus the semigroup $\mathbb{S}_6$ contains 8 groups: $\mathbb{G}_3$, $\mathbb{G}_4$, $\mathbb{G}_6$, $\mathbb{G}_7$, $\mathbb{G}_8$, $\mathbb{G}_9$, $\mathbb{G}_{10}$ and $\mathbb{G}_{11}$; and 10 semigroups: $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_3$, $\mathbb{I}_4$, $\mathbb{S}_1$, $\mathbb{S}_5$, $\mathbb{S}_6$, $\mathbb{S}_7$, $\mathbb{S}_8$ and $\mathbb{S}_9$. In amount, $\mathbb{S}_6$ contains eighteen semigroups.

Remaining groups with two generators in $\mathbb{S}_2$

Monoids

- $< \sigma_2, \sigma_6 >= \{\sigma_2, \sigma_6, \sigma_{10}\} = \mathbb{M}_2 = \mathbb{A}[\mathbb{M}_5]$;
- $< \sigma_2, \sigma_{10} >= \{\sigma_2, \sigma_{10}\} = \mathbb{M}_3 = \mathbb{A}[\mathbb{M}_4]$;
- $< \sigma_5, \sigma_{13} >= \{\sigma_5, \sigma_{13}\} = \mathbb{M}_4 = \mathbb{A}[\mathbb{M}_3]$;
- $< \sigma_5, \sigma_9 >= \{\sigma_5, \sigma_9, \sigma_{13}\} = \mathbb{M}_5 = \mathbb{A}[\mathbb{M}_2]$.

Non-monoids

- $< \sigma_2, \sigma_7 >= \{\sigma_2, \sigma_7, \sigma_{10}\} = \mathbb{I}_5 = \mathbb{A}[\mathbb{I}_8]$;
- $< \sigma_2, \sigma_8 >= \{\sigma_2, \sigma_8, \sigma_{10}\} = \mathbb{I}_6 = \mathbb{A}[\mathbb{I}_7]$;
- $< \sigma_5, \sigma_7 >= \{\sigma_5, \sigma_7, \sigma_{13}\} = \mathbb{I}_7 = \mathbb{A}[\mathbb{I}_6]$;
- $< \sigma_5, \sigma_8 >= \{\sigma_5, \sigma_8, \sigma_{13}\} = \mathbb{I}_8 = \mathbb{A}[\mathbb{I}_5]$.

- $<\sigma_2,\sigma_{11}>=<\sigma_2,\sigma_6,\sigma_7>=\{\sigma_2,\sigma_6,\sigma_7,\sigma_{10},\sigma_{11}\}=\{\sigma_2\}\cup\mathbb{S}_7$.
- $<\sigma_2,\sigma_{12}>=<\sigma_2,\sigma_6,\sigma_8>=\{\sigma_2,\sigma_6,\sigma_8,\sigma_{10},\sigma_{12}\}=\{\sigma_2\}\cup\mathbb{S}_8$.
- $<\sigma_2,\sigma_{13}>=<\sigma_2,\sigma_7,\sigma_8>=\{\sigma_2,\sigma_7,\sigma_8,\sigma_{10},\sigma_{13}\}=\{\sigma_2\}\cup\mathbb{S}_1$.
- $<\sigma_2,\sigma_9>=\{\sigma_2\}\cup\mathbb{S}_6$.

Sets of generators: $<\sigma_2,\sigma_6,\sigma_{13}>$, $<\sigma_2,\sigma_7,\sigma_{12}>$, $<\sigma_2,\sigma_8,\sigma_{11}>$, $<\sigma_2,\sigma_{11},\sigma_{12}>$, $<\sigma_2,\sigma_{11},\sigma_{13}>$ and $<\sigma_2,\sigma_{12},\sigma_{13}>$ are minimal in $\{\sigma_2\}\cup\mathbb{S}_6$.

- $\mathbb{A}[<\sigma_2,\sigma_{11}>] =<\sigma_5,\sigma_{12}>=<\sigma_5,\sigma_9,\sigma_8>= \{\sigma_5,\sigma_9,\sigma_8,\sigma_{13},\sigma_{12}\} = \{\sigma_5\}\cup\mathbb{S}_5$;
- $\mathbb{A}[<\sigma_2,\sigma_{12}>] =<\sigma_5,\sigma_{11}>=<\sigma_5,\sigma_9,\sigma_7>= \{\sigma_5,\sigma_9,\sigma_7,\sigma_{13},\sigma_{11}\} = \{\sigma_5\}\cup\mathbb{S}_9$;
- $\mathbb{A}[<\sigma_2,\sigma_{13}>] =<\sigma_5,\sigma_{10}>=<\sigma_5,\sigma_8,\sigma_7>= \{\sigma_5,\sigma_8,\sigma_7,\sigma_{13},\sigma_{10}\} = \{\sigma_5\}\cup\mathbb{S}_1$.
- $\mathbb{A}[<\sigma_2,\sigma_9>] =<\sigma_5,\sigma_6>= \{\sigma_5\}\cup\mathbb{S}_6= <\sigma_5,\sigma_9,\sigma_{10}> = <\sigma_5,\sigma_8,\sigma_{11}> = <\sigma_5,\sigma_7,\sigma_{12}> = <\sigma_5,\sigma_{12},\sigma_{11}> = <\sigma_5,\sigma_{12},\sigma_{10}> = <\sigma_5,\sigma_{11},\sigma_{10}>$.

- $< \sigma_3, \sigma_6 >=< \sigma_3, \sigma_{10} >= \{\sigma_3, \sigma_6, \sigma_7\ \sigma_{10}, \sigma_{11}\} = \{\sigma_3\} \cup \mathbb{S}_7$;
- $< \sigma_3, \sigma_9 >=< \sigma_3, \sigma_{13} >= \{\sigma_3, \sigma_7, \sigma_9\ \sigma_{11}, \sigma_{13}\} = \{\sigma_3\} \cup \mathbb{S}_9$;
- $< \sigma_3, \sigma_8 >=< \sigma_3, \sigma_{12} >= \{\sigma_3\} \cup \mathbb{S}_6$.
- $\mathbb{A}[< \sigma_3, \sigma_6 >] =< \sigma_4, \sigma_9 >=< \sigma_4, \sigma_{13} >= \{\sigma_4, \sigma_8, \sigma_9\ \sigma_{12}, \sigma_{13}\} = \{\sigma_4\} \cup \mathbb{S}_5$;
- $\mathbb{A}[< \sigma_3, \sigma_9 >] =< \sigma_4, \sigma_6 >=< \sigma_4, \sigma_{10} >= \{\sigma_4, \sigma_6, \sigma_8\ \sigma_{10}, \sigma_{12}\} = \{\sigma_4\} \cup \mathbb{S}_8$;
- $\mathbb{A}[< \sigma_3, \sigma_8 >] =< \sigma_4, \sigma_7 >=< \sigma_4, \sigma_{11} >= \{\sigma_4\} \cup \mathbb{S}_6$.

Additionally,

- $\{\sigma_3\} \cup \mathbb{S}_6 =< \sigma_3, \sigma_6, \sigma_9 >=< \sigma_3, \sigma_6, \sigma_{13} >=< \sigma_3, \sigma_9, \sigma_{10} >=< \sigma_3, \sigma_{10}, \sigma_{13} >$;
- $\{\sigma_4\} \cup \mathbb{S}_6 =< \sigma_4, \sigma_6, \sigma_9 >=< \sigma_4, \sigma_9, \sigma_{10} >=< \sigma_4, \sigma_6, \sigma_{13} >=< \sigma_4, \sigma_{10}, \sigma_{13} >$.

- $<\sigma_2,\sigma_3>=\{\sigma_2,\sigma_3,\sigma_6,\sigma_7,\sigma_{10}\,\sigma_{11}\}=\{\sigma_2,\sigma_3\}\cup\mathbb{S}_7$.
- $<\sigma_2,\sigma_3,\sigma_8>=<\sigma_2,\sigma_3,\sigma_9>=<\sigma_2,\sigma_3,\sigma_{12}>=<\sigma_2,\sigma_3,\sigma_{13}>=\{\sigma_2,\sigma_3\}\cup\mathbb{S}_6$.
- $\mathbb{A}[<\sigma_3,\sigma_2>]=<\sigma_4,\sigma_5>=\{\sigma_4,\sigma_5,\sigma_8,\sigma_9,\sigma_{12}\,\sigma_{13}\}=\{\sigma_4,\sigma_5\}\cup\mathbb{S}_5$.
- $\mathbb{A}[<\sigma_3,\sigma_2,\sigma_8>]=<\sigma_4,\sigma_5,\sigma_7>=<\sigma_4,\sigma_5,\sigma_6>=<\sigma_4,\sigma_5,\sigma_{11}>=<\sigma_4,\sigma_5,\sigma_{10}>=\{\sigma_4,\sigma_5\}\cup\mathbb{S}_6$.
- $<\sigma_2,\sigma_4>=\{\sigma_2,\sigma_4,\sigma_6,\sigma_8,\sigma_{10}\,\sigma_{12}\}=\{\sigma_2,\sigma_4\}\cup\mathbb{S}_8$;
- $<\sigma_2,\sigma_4,\sigma_7>=<\sigma_2,\sigma_4,\sigma_9>=<\sigma_2,\sigma_4,\sigma_{11}>=<\sigma_2,\sigma_4,\sigma_{13}>=\{\sigma_2,\sigma_4\}\cup\mathbb{S}_6$;

- $\mathbb{A}[<\sigma_4,\sigma_2>] =<\sigma_3,\sigma_5>= \{\sigma_3,\sigma_5,\sigma_7,\sigma_9,\sigma_{11}\,\sigma_{13}\} = \{\sigma_3,\sigma_5\}\cup\mathbb{S}_9$;
- $\mathbb{A}[<\sigma_4,\sigma_2,\sigma_7>] =<\sigma_3,\sigma_5,\sigma_8>=<\sigma_3,\sigma_5,\sigma_6>=<\sigma_3,\sigma_5,\sigma_{12}>=<\sigma_3,\sigma_5,\sigma_{10}>= \{\sigma_3,\sigma_5\}\cup\mathbb{S}_6$;
- $<\sigma_2,\sigma_5>= \{\sigma_2,\sigma_5,\sigma_7,\sigma_8,\sigma_{10}\,\sigma_{13}\} = \{\sigma_2,\sigma_5\}\cup\mathbb{S}_1$;
- $<\sigma_2,\sigma_5,\sigma_6>=<\sigma_2,\sigma_5,\sigma_9>=<\sigma_2,\sigma_5,\sigma_{11}>=<\sigma_2,\sigma_5,\sigma_{12}>= \{\sigma_2,\sigma_5\}\cup\mathbb{S}_6$;
- $<\sigma_2,\sigma_3,\sigma_5>= \{\sigma_2,\sigma_3,\sigma_5\}\cup\mathbb{S}_6$;
- $\mathbb{A}[<\sigma_5,\sigma_3,\sigma_2>] =<\sigma_2,\sigma_4,\sigma_5>= \{\sigma_2,\sigma_4,\sigma_5\}\cup\mathbb{S}_6$.

Thus, the algorithm which enumerates all semigroups (without σ_0) is ready. One can check that it enumerates four (4) monoids with σ_0 and fifty seven (57) semigroups without σ_0. Thus, the monoid $\mathbb{M}$ contains $118 = 4 + 57 \cdot 2$ semigropus.

Exercises easy!?

Let X be a extremally disconnected space, then $\sigma_6 = \sigma_{12}$, $\sigma_7 = \sigma_{13}$, $\sigma_8 = \sigma_{10}$ and $\sigma_9 = \sigma_{11}$. How many errors does contain the following table of compositions of Kuratowski operations?

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
σ_1	σ_0	σ_4	σ_5	σ_2	σ_3	σ_8	σ_9	σ_6	σ_7
σ_2	σ_3	σ_2	σ_3	σ_6	σ_7	σ_6	σ_7	σ_8	σ_9
σ_3	σ_2	σ_6	σ_7	σ_2	σ_3	σ_8	σ_9	σ_6	σ_7
σ_4	σ_5	σ_4	σ_5	σ_8	σ_9	σ_8	σ_9	σ_6	σ_7
σ_5	σ_4	σ_8	σ_9	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9
σ_6	σ_7	σ_6	σ_7	σ_8	σ_9	σ_8	σ_9	σ_6	σ_7
σ_7	σ_6	σ_8	σ_9	σ_6	σ_7	σ_6	σ_7	σ_8	σ_9
σ_8	σ_9	σ_8	σ_9	σ_6	σ_7	σ_6	σ_7	σ_8	σ_9
σ_9	σ_8	σ_6	σ_7	σ_8	σ_9	σ_8	σ_9	σ_6	σ_7

If you believe that it does not have errors, then find (describe) and count all semigroups in monoids $\mathbb{M}$ for X. There are 70 semigroups!?

Have I a several years to crack (using pencil and paper only) the problem: Describe monoids generated by words

$$"\sigma_i \circ f^{-1} \circ \sigma_j \circ f \circ \sigma_k" \text{ or } "\sigma_i \circ f \circ \sigma_j \circ f^{-1} \circ \sigma_k",$$

where σ_n are Kuratowski operations?