KATOWICE "PATTERN" FOR THE QUESTIONS INSPIRED BY Czech mathematicians (personal perspective)

by

Szymon Plewik

(*Katowice*)

THE WINTER SCHOOL IN ABSTRACT ANALYSIS
Section Set Theory & Topology
January 28 - February 4, (2012).

Generalizing previously known property of nowhere separable metric spaces, W. Kulpa and A. Szymański [Bull. Acad. Polon. Sci. 25 (1977)] have stated as follows. For inspiration, compare the paper by P. Štěpánek and P. Vopěnka (1967).

Kulpa-Szymański Theorem (1977)

Let X be a topological space with a π-basis $\bigcup \Theta$. Suppose λ is an (the least) infinite cardinal such that $\Theta = \{H_\alpha : \alpha < \lambda\}$, where each H_α consists of pairwise disjoint sets. If any non-empty open set contains λ^+ pairwise disjoint open sets, then there exists a family $\{D_\beta : \beta < \lambda^+\}$ of nowhere dense sets covering X.

For metric nowhere separable spaces, i.e. a metric space such that no open and non-empty subset is separable, $\lambda = \omega_0$, so any such space is an union of an increasing family (of size ω_1) of nowhere dense sets.

Non-trivial applications of Kulpa-Szymański Theorem is in the paper "*The space of ultrafilters on N covered by nowhere dense sets*" by B. Balcar, J. Pelant and P. Simon [Fund. Math. (1980)]. The authors mention a conversation (about the Novak number) with J. Mioduszewski at the V-th Winter School in Štefanová. This was the first article that I had to read and to set forth at the seminar in Katowice. The nicest result of this paper is:

Base Matrix Theorem (Lemma 2.11)

There exists a collection $\Theta = \{H_\alpha : \alpha < h\}$ of MAD families such that $\bigcup \Theta$ is dense in the poset $([\omega]^\omega, \subseteq^*)$ and each H_α refines H_β for $\beta < \alpha$ (and h is the least possible cardinal).

Together with Kulpa-Szymański Theorem, this assertion was used to estimate covering and additivity numbers of nowhere dense subsets in the space of ultrafilters on ω.

Motivated by the Balcar, Pelant and Simon paper, we consider (I am advocating in Katowice) the following pattern for questions.

- Suppose that we have is an ideal (q), which is associated to some forcing conditions - one of these at the paper "*Strolling through paradise*" by J. Brendle [Fund. Math. (1995)]
- Is there a topology on $X = \bigcup(q)$, which satisfies the assumptions Kulpa-Szymański Theorem such that (q) is a family of nowhere dense sets in this topology?

If EXISTS A SUCH TOPOLOGY and λ is the least possible cardinal, then

$$\omega_0 \leq add((q)) = \lambda \leq cov((q)) \leq add((q))^+ = \lambda^+.$$

When trees are elements of a poset $(\mathbb{Q}, <)$, then one can consider following ideals associated to this poset.

- $(q^0) = \{Y \subseteq X^\omega : \forall_{p \in \mathbb{Q}} \exists_{q<p} \ [q] \cap S = \emptyset\}$;
- $(q^1) = \{S \subseteq \mathbb{Q} : \forall_{p \in \mathbb{Q}} \exists_{q<p} \ \{y : y \leq q\} \cap S = \emptyset\}$;
- $(q^2) = \{S \subseteq \mathbb{Q}^* : \forall_{p \in \mathbb{Q}} \exists_{q<p} \ q^* \cap S = \emptyset\}$, where $\mathbb{Q}^*$ is the family of all maximal centered subfamilies of $\mathbb{Q}$ (in fact, the space of all ultrafilter) and $q^* = \{\mu \in \mathbb{Q}^* : q \in \mu\}$.

Above, each tree $p \in \mathbb{Q}$ is contained in Seq_X, i.e. in the family which consists of finite sequences of elements of X, and its body $[p]$ (all infinite branches) is a perfect subset of the product space X^ω.

For the MATHIAS FORCING the ideal (q^0) is denoted as (r^0). It is the family of nowhere Ramsey sets or the family of nowhere dense sets with respect to the Ellentuck topology. Base Tree Theorem implies [Me, Fund. Math. (1987)]

$$add((r^0)) = cov((r^0)) = h.$$

Also, by the definitions (identifying a set with its characteristic function) and the above estimate we obtain

$$add((r^1)) = cov((r^1)) = h.$$

As well, $h < \mathfrak{c}$ implies $h = add((r^2)) \leq cov((r^2)) = h^+$, see B. Balcar, J. Pelant and P. Simon, Fund. Math. (1980). Kulpa-Szymański Theorem is essential for evidence relating to (r^2), only.

Michał Machura [Tatra Mountains Mathematical Publications (2004)], presented at 32-th (?) Winter School, too], consider the following poset:

Partial order on the family of continuous functions from a topological space X into $[\omega]^\omega$ is defined as follows

$$f \subseteq *g \text{ if and only, if } f(x) \subseteq^* g(x) \text{ for any } x \in X.$$

For this poset a variant Base Tree Theorem was established, too.

I will add yet, that Machura's paper used tricks and facts described in a few articles (by several authors, but not from Katowice). For example, ones by S. Shelah and O. Spinas: Fund. Math. 158 (1999); Trans. Amer. Math. Soc. (1999).

For the SILVER FORCING the ideal (q^0) is denoted as (v^0). To describe it accurately, we need some definitions. Anna Wojciechowska discussed them three years ago at Winter School in Hejnice - a published paper by P. Kalemba, Me and A. Wojciechowska is in Cent. Eur. J. Math. 6 (2008).

Thus, conditions could be represented by the so-called segments, that is, sets of the form

$$\langle A, B \rangle = \{X \in [\omega]^\omega : A \subseteq X \subseteq B\},$$

where $A \in [\omega]^\omega$, $A \subset B$ and $B \setminus A \in [\omega]^\omega$. Inclusion gives the partial order.

A set $X \subseteq 2^\omega$ belongs to (v^0) whenever for every segment $\langle A, B \rangle$ there exists a segment $\langle C, D \rangle \subseteq \langle A, B \rangle$ such that

$$X \cap \langle C, D \rangle = \emptyset.$$

Similar as with the family of nowhere Ramsey sets a variant of Base Tree Theorem holds, but with respect to so called $*-$segments, that is, sets of the form

$$\langle A, B\rangle^* = \{X \in [\omega]^\omega : A \subseteq^* X \subseteq^* B\},$$

where $\langle A, B\rangle$ runs over segments.

Two years ago, P. Kalemba discussed [at Winter School in Hejnice] the possible generalizations of the ideals (v^0): trees with bodies in X^ω (for finite X). Exhausting (?) the possibilities offered by the replacement of trees (perfect sets) onto some countable sum of their shifts. In the paper by Me and P. Kalemba [Cent. Eur. J. Math. (2010)] a versions of Base Tree Theorem are proven and Kulpa-Szynański Theorem gives (below (v^0) denotes some its generalization, too)

$$\omega_1 \leq add((v^0)) \leq cov((v^0)) \leq add((v^0))^+.$$

In fact, if we have a family $\{\mathbb{P}_\alpha : \alpha < \lambda \leq \mathfrak{c}\}$ of posets such that each one satisfies Base Tree Theorem, then we can consider each ideal associated to $\mathbb{P}_\alpha$ as the ideals of nowhere dense subsets with respect to a topology. Than, one can check that the product of these topologies satisfies Base Tree Theorem, too. But the height of a such tree can be less than h, in others words: we can a natural class of topologies, which satisfies Baire Theorem, but their Cartesian products does not preserve Baire Numbers (understand as the additivity - first category sets are nowhere dense in these cases - or covering numbers).

The trick with the replacement of a tree's body by the countable sum of their shifts, i.e.

$$[p] \Longrightarrow [p]^* = \{f \in X^\omega : f =^* g \text{ and } g \in [p]\},$$

does not work for many other forcing conditions. Often, it leads to not σ-closed posets.

Let example clarify Base Tree Theorem schema:

Consider the paper "*Sacks forcing, Laver forcing, and Martin's axiom*" [Arch. Math. Logic 31 (1992)] by H. Judah, A. W. Miller and S. Shelah. Here are defined a poset as follows.

- Let $\overline{A} = \langle A_s \in [\omega]^\omega : s \in \omega^{<\omega} \rangle$ and Q be the collections of all $\overline{A}$;
- For $\overline{A}, \overline{B} \in Q$ define $\overline{A} \subseteq^* \overline{B}$ if and only, if $A_s \setminus B_s$ is finite for any $s \in \omega^{<\omega}$;
- Let $p_s(\overline{A})$ be the unique Laver tree such that the root of $p_s(\overline{A})$ is s and for every $t \supseteq s$ with $t \in p_s(\overline{A})$ we have that $split(p, t) = A_t$.

The poset $(Q, \subseteq^*)$ fulfills :

- The set Q has the cardinality continuum, i.e. $|Q| = \mathfrak{c}$;
- The poset $(Q, \subseteq^*)$ is separative, i.e. if $\overline{A}, \overline{B} \in Q$ and does not hold $\overline{A} \subseteq^* \overline{B}$, then there is $\overline{C} \in Q$ such that $\overline{C} \subseteq^* \overline{A}$ and B is not comparable with C;
- $\langle \Omega_s = \omega : s \in [\omega]^{<\omega} \rangle$ is the greatest element in Q;
- The poset $(\{\overline{B} \in Q : B_s \subseteq A_s \text{ for } s \in \omega^{<\omega}\}, \subseteq^*)$ is isomorphic with $(Q, \subseteq^*)$ for each $\overline{A} \in Q$;
- The poset $(Q, \subseteq^*)$ is σ-closed, i.e. decreasing and countable sequences are bounded;
- There are continuum many pairwise not comparable elements below each $\overline{A} \in Q$.

The above properties implies a version of Base Tree Theorem for $(Q, \subseteq^*)$. Kulpa-Szymański Theorem works with ideals (q^1) or (q^2) as (almost) in the Balcar, Pelant and Simon paper. But, problems is with counterparts of (q^0).

In particular, let (l^0) be the ideal associated to the LAVER FORCING [$X \in (l^0)$ if and only, if for every Laver tree p there exists a Laver tree b such that $[b] \subseteq [p]$ and $X \cap [b] = \emptyset$]. To obtain $(l^0) = (q^0)$ one should exchange the partial order. If we put $\overline{A} \prec \overline{B}$ whenever $A_s \subseteq B_s$ for all but finite many $s \in \omega^{<\omega}$ [compare with the article "*On tree ideals*" by M. Goldstern, M. Repický, S. Shelah and O. Spinas in Proc. Amer. Math. Soc. 123 (1995)], then the poset $(\mathbb{Q}, \prec)$ is not separative. A suitable version of the Base Matrix Tree for non-separative poset is needed. The same holds with the ideals associated with MILLER FORCING, i.e. (m^0), or SAKS FORCING, i. e. (s^0).

Currently, Anna Wojciechowska is preparing a doctoral thesis with the above topics. A non-separative version of Base Tree Theorem for $(\mathbb{Q}, \prec)$ is ready. It contains the concepts associated with the separative modification.

Countable sum of a tree shifts in the above mentioned trick, is replaced by another type of F_σ sets, i.e.

$$[p] \Longrightarrow \bigcup\{[p_s(\overline{A})] : s \in \omega^{<\omega}\}.$$

It gives $(l^0) = (q^0)$. I am convinced that counterparts for (m^0) and (s^0) will be included in the Anna's dissertation, too. This requires some time before it finally is prepared in the form of publications...

Explanations

1. Sentence written in blue are saying about facts;
2. Sentence written in red are saying about observations;
3. Sentence written in violet are saying about personal observations;
4. White or black (?) are background colors.

THANK YOU