Variations of Separability

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Winter School in Abstract Analysis
section Set Theory

Outline

properties stating that a space has a small dense set

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Evolution

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separable

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- *d*-separable: there is a σ-discrete dense set

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separable

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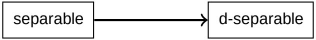
- Kurepa: property K_0

Evolution



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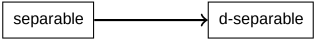

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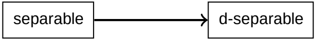

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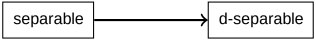

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- *nwd*-separable:there is a σ-nwd dense set
  (a dense set which is the countable union of nowhere dense subsets)

Evolution

d-separable: there is a σ-discrete dense set

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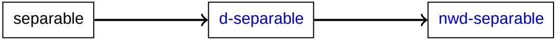

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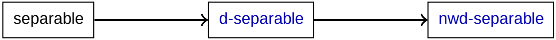

So-So-Sp:

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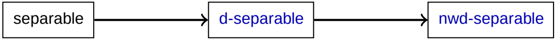

So-So-Sp:

- products of nwd-separable spaces are nwd-separable

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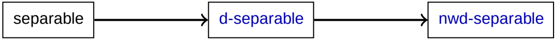

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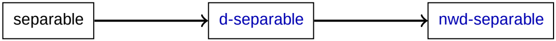

So-So-Sp:

- products of nwd-separable spaces are nwd-separable
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- Is there a non-*nwd*-separable space with a *nwd*-separable square?

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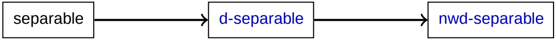

So-So-Sp:

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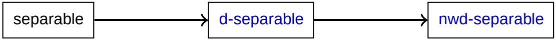
- X^ω is *nwd*-separable
- Is there a non-*nwd*-separable space with a *nwd*-separable square?
- there is a compact nwd-separable space which is not d-separable: $X = \omega^* \times D(2)^\omega$

Selection Principles



Selection Principles

separable → d-separable → nwd-separable

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- $G_1(\mathcal{D}, \mathcal{D})$: I. picks a dense D_0, II picks $x_0 \in D_0$, I. picks a dense D_1, II picks $x_1 \in D_1$, etc

  II wins iff $\{x_0, x_1, \ldots\}$ is dense.

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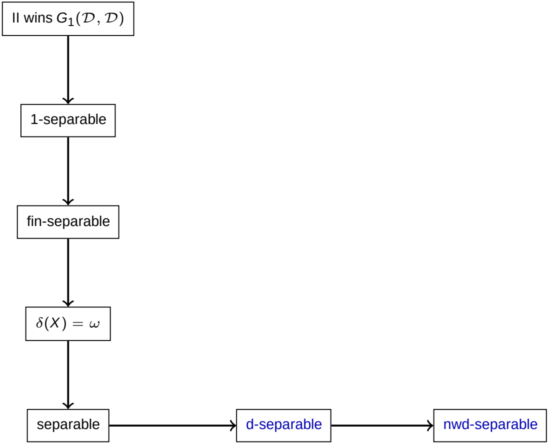
- II wins $G_1(\mathcal{D}, \mathcal{D}) \implies$ 1-separable $\implies$ fin-separable $\implies \delta(X) = \omega$

Selection principles



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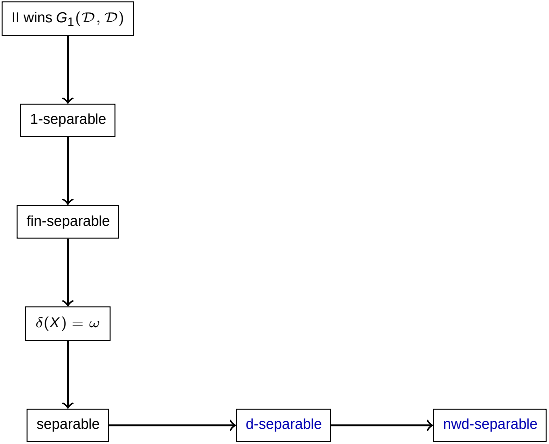

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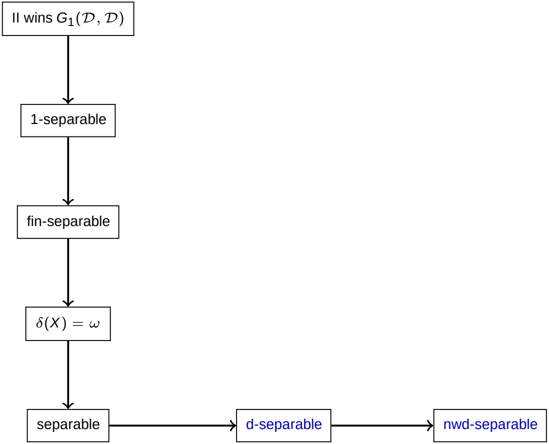
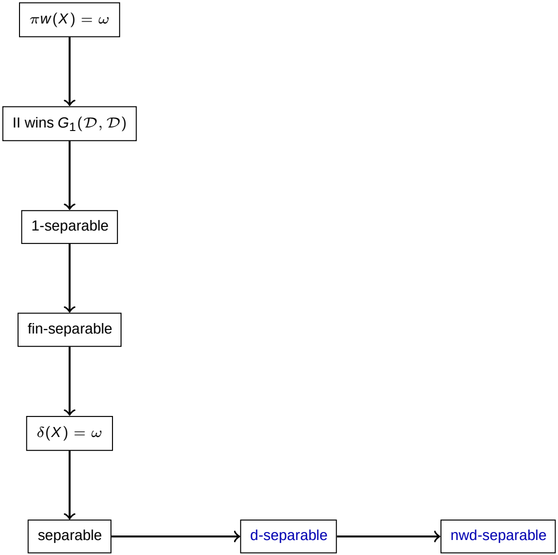


- How to guarantee that II wins $G_1(\mathcal{D}, \mathcal{D})$?
- $\pi w(X) = \omega$: if $\{U_n : n \in \omega\}$ is a π-base, II picks $x_n \in D_n \cap U_n$.

A positive result



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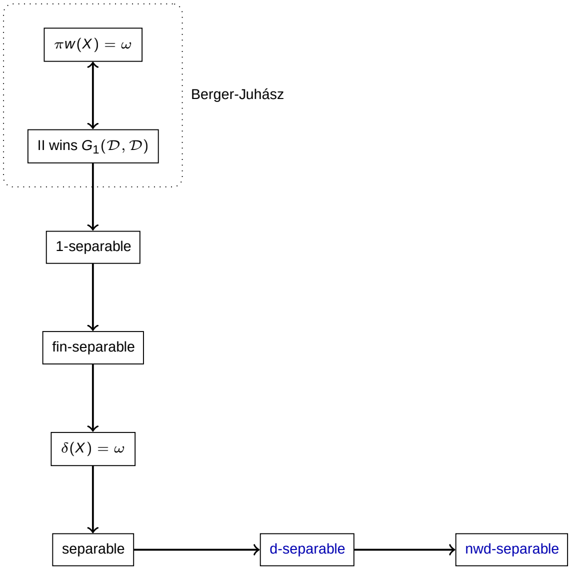
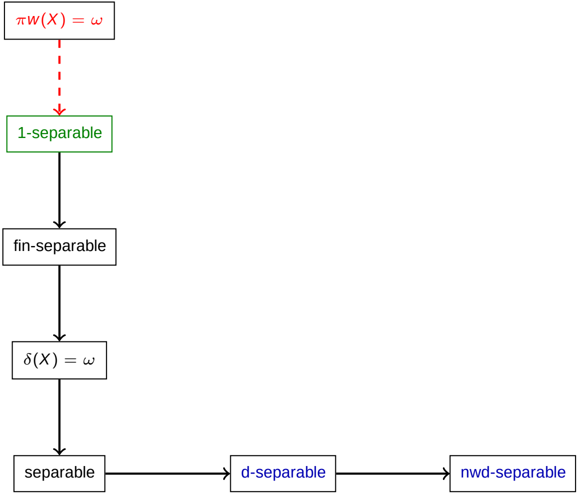


Separations of properties



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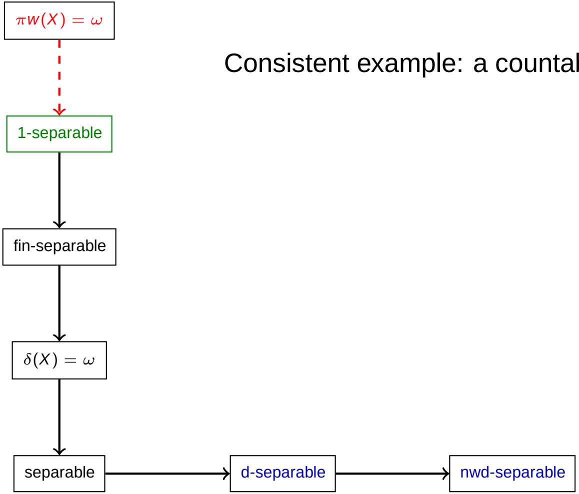
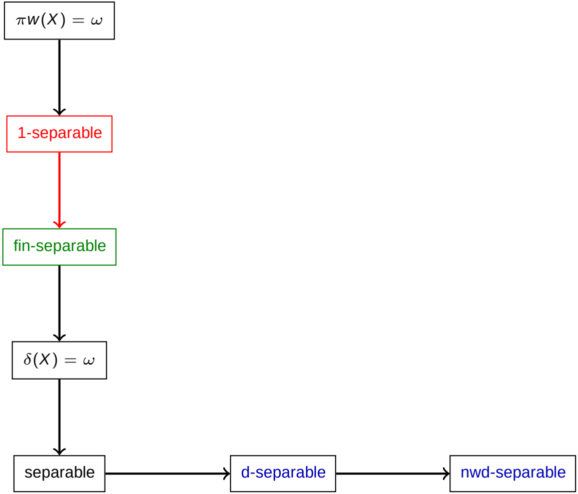


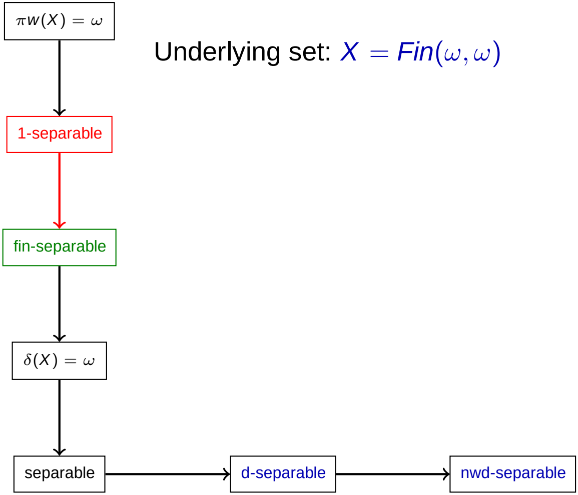
Consistent example: a countable HFD

Separations of properties



Separations of properties

Underlying set: $X = Fin(\omega, \omega)$



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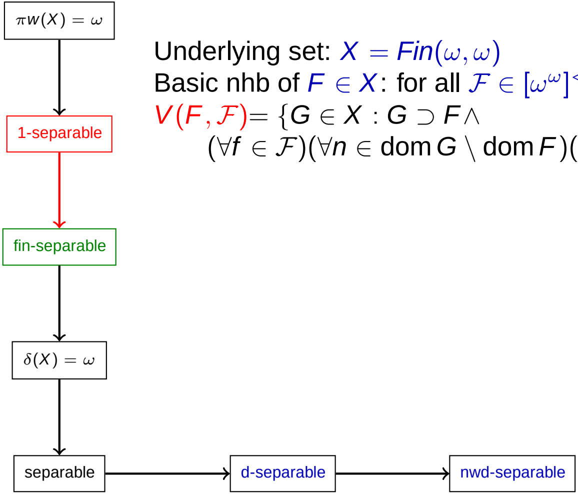
$$V(F, \mathcal{F}) = \{G \in X : G \supset F \wedge (\forall f \in \mathcal{F})(\forall n \in \operatorname{dom} G \setminus \operatorname{dom} F)(G(n) \neq f(n))\}$$



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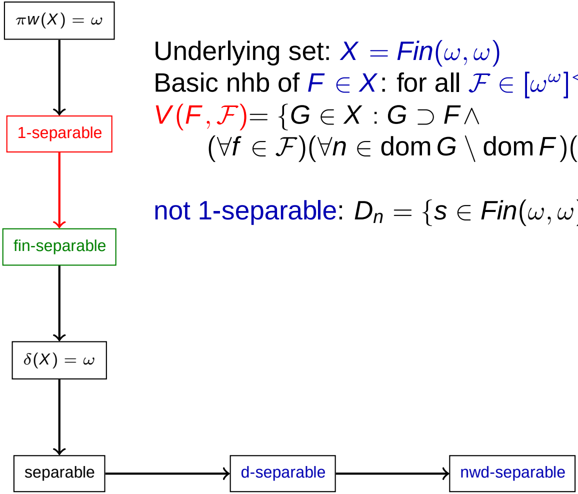

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not 1-separable: $D_n = \{s \in Fin(\omega, \omega) : |s| > n\}$

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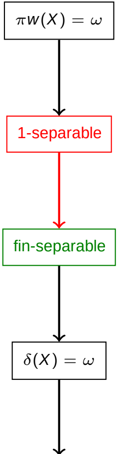

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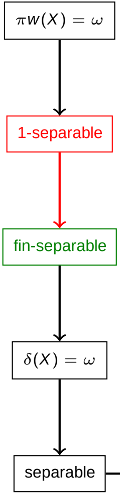

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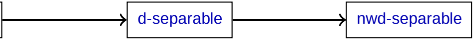
Define $f : \omega \to \omega$ by $f(n) = s_n(n)$



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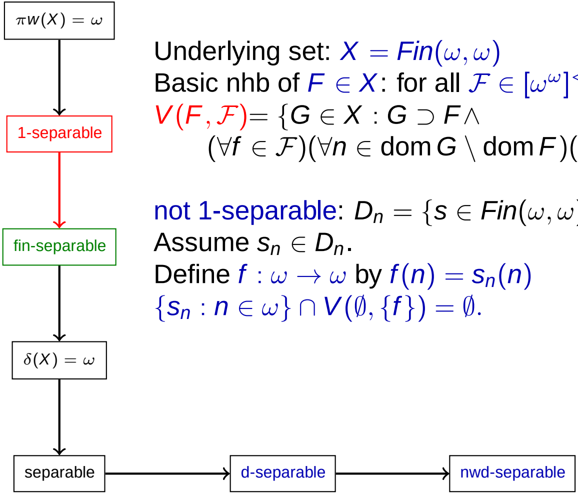

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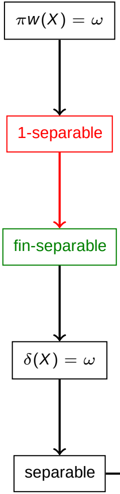

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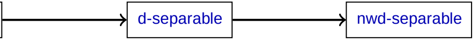
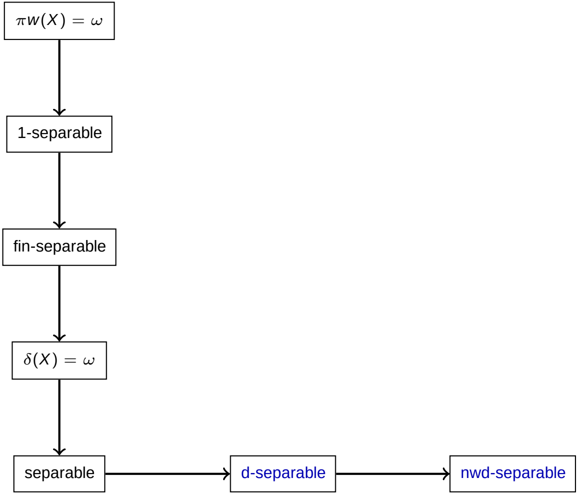
fin-separable: eventually different forcing



Classical positive results



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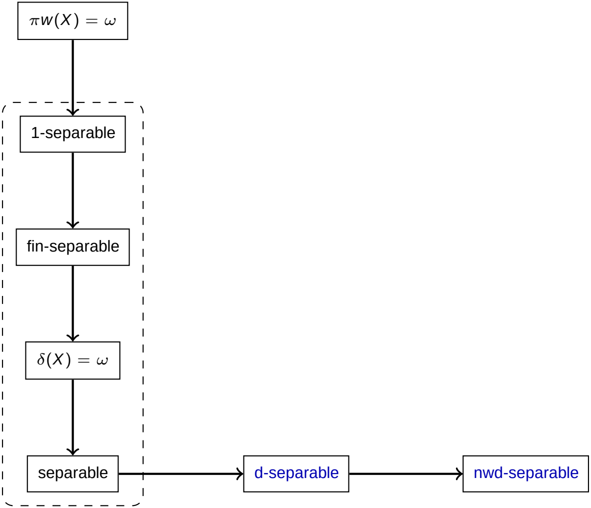
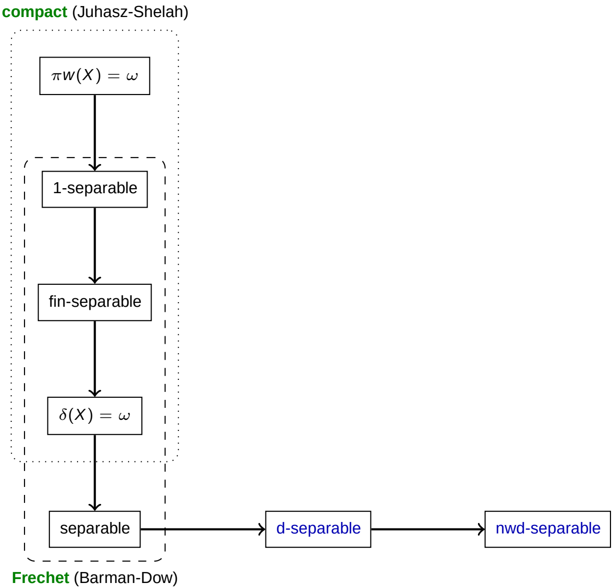
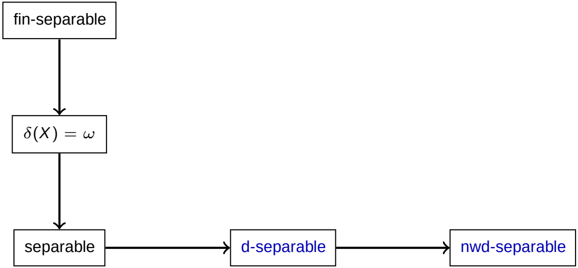


Frechet (Barman-Dow)

Classical positive results



Selection Principles



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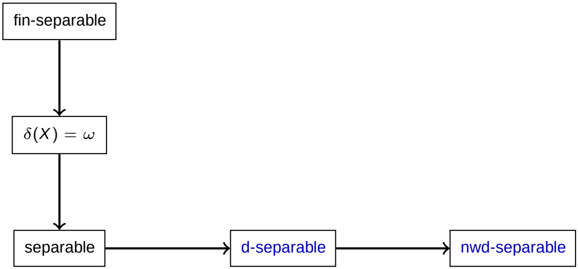

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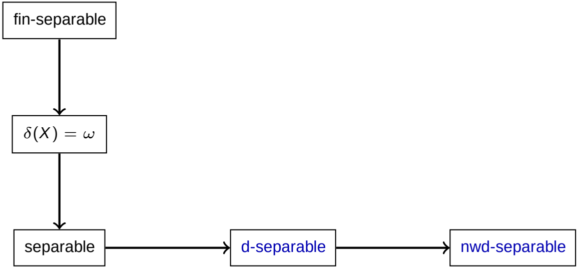

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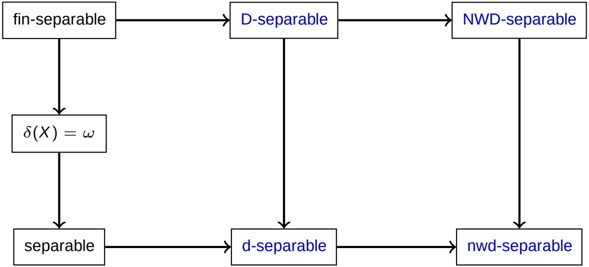

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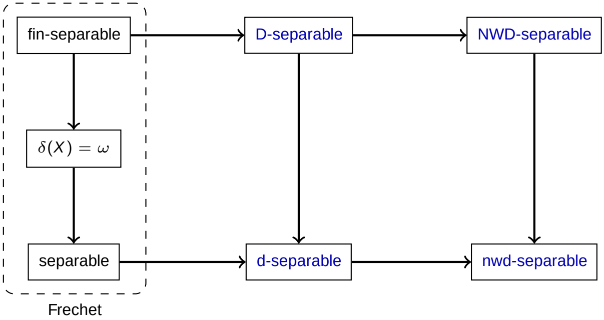

Selection Principles

X is D-separable iff
$\forall \{D_n\}_{n\in\omega} \subset \mathcal{D}\ \exists F_n \subset D_n$ discrete $\bigcup\{F_n : n \in \omega\} \in \mathcal{D}$
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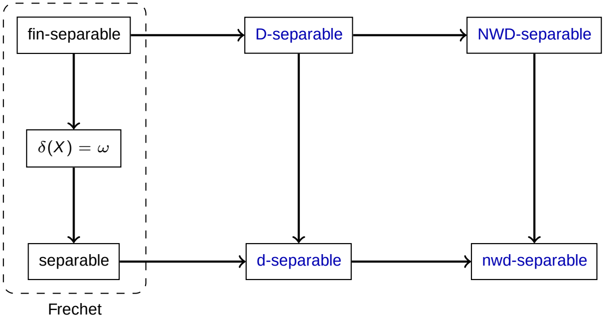
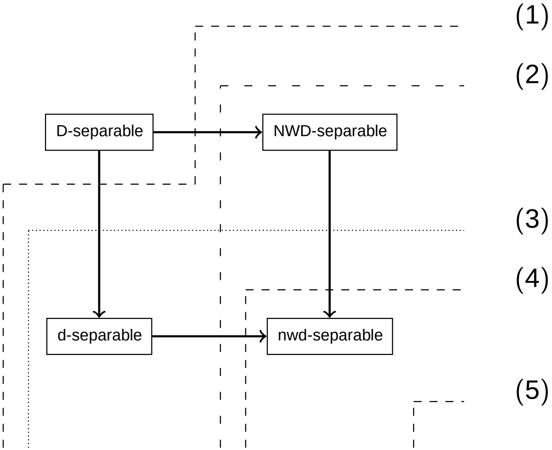


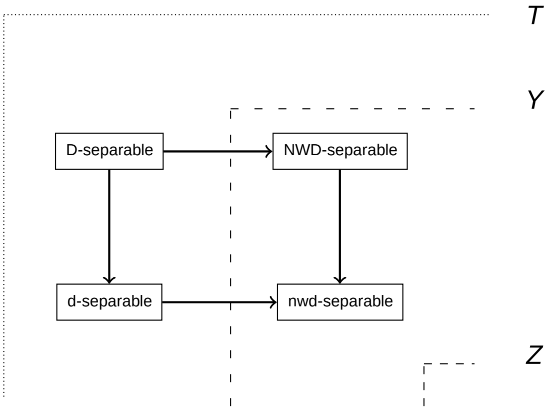
No other implication for first countable spaces

A separation theorem



A separation theorem

So-So-Sp: Con($\exists$ first countable X and X has a partition $T \cup^* Y \cup^* Z$ into uncountable dense subspaces s.t. X is left-separated in type ω_1,



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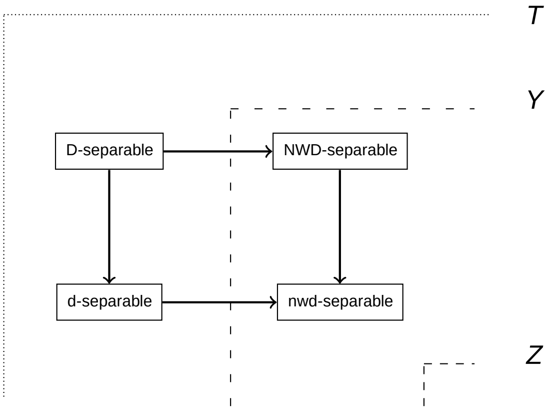

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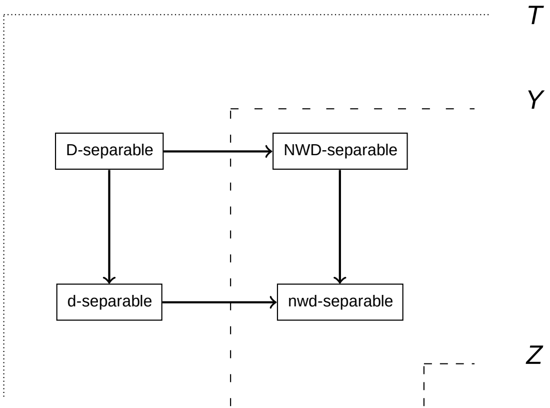
So-So-Sp: Con($\exists$ first countable X and X has a partition $T \cup^* Y \cup^* Z$ into uncountable dense subspaces s.t. X is left-separated in type ω_1, (1) T is D-separable; (2) Y is NWD-separable, but not d-separable;



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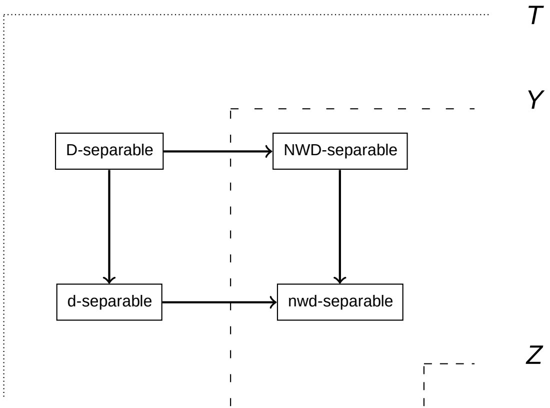

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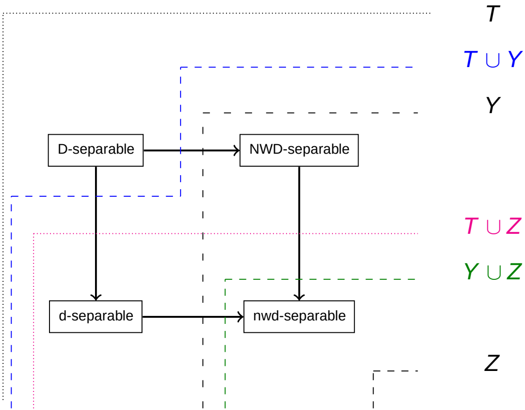

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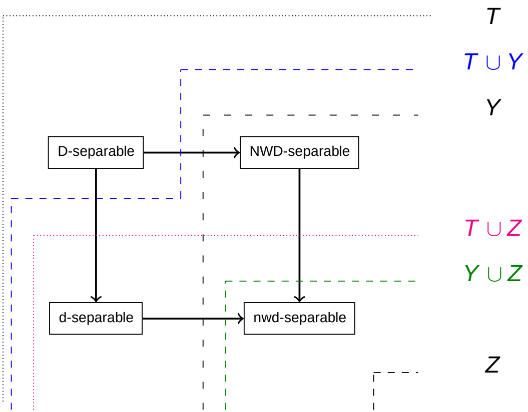

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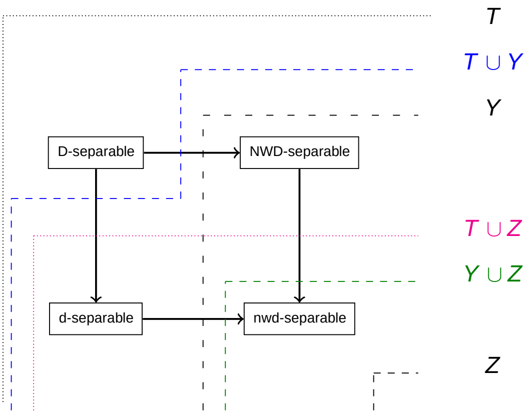

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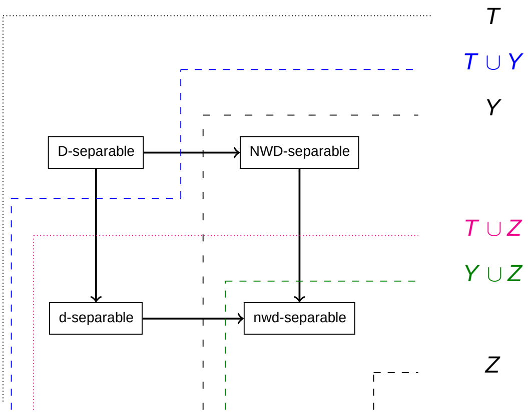
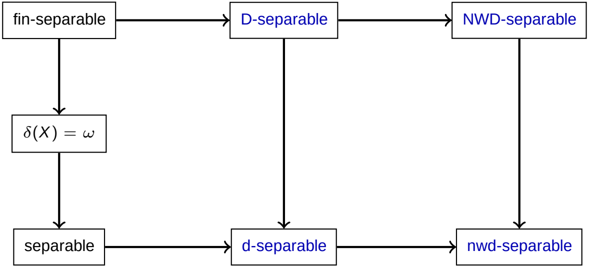


Separation theorems in ZFC



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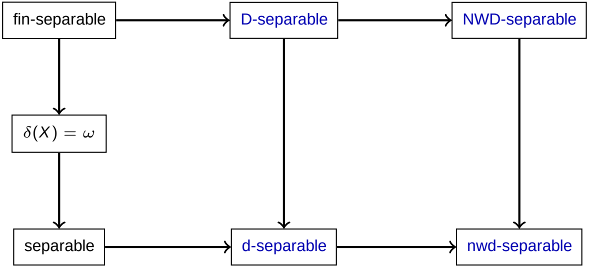

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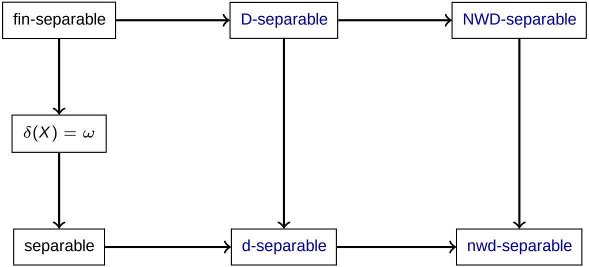

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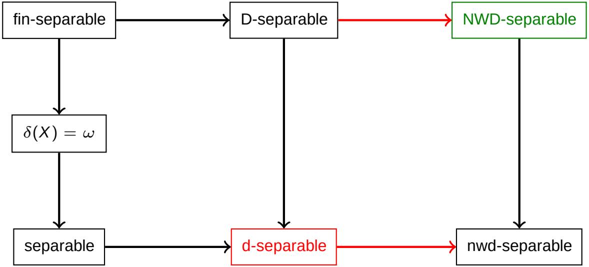
Separation theorems in ZFC



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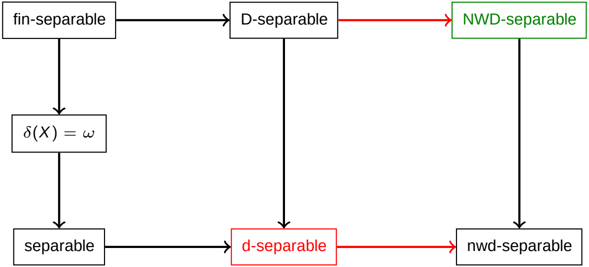
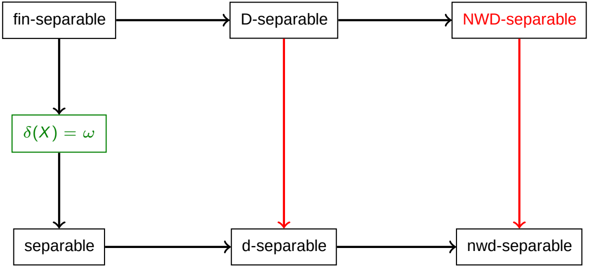


- If $Y = X \times \mathbb{Q}$, where X is the G_δ topology on $D(2)^{\omega_1}$, then Y is NWD-separable, but not d-separable.

Separation theorems in ZFC



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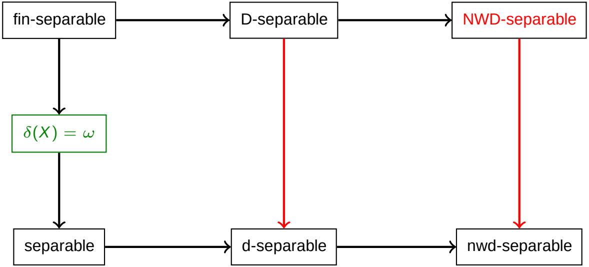

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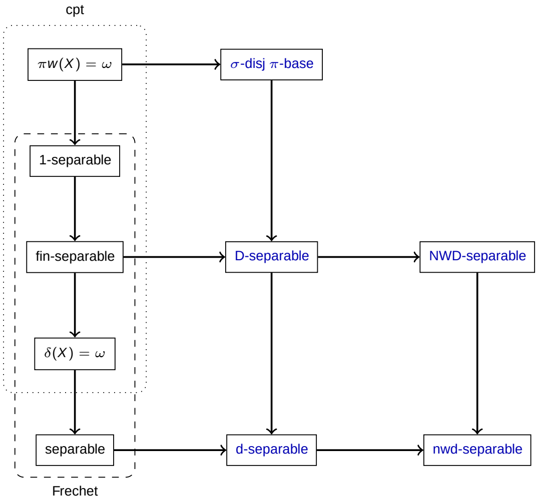
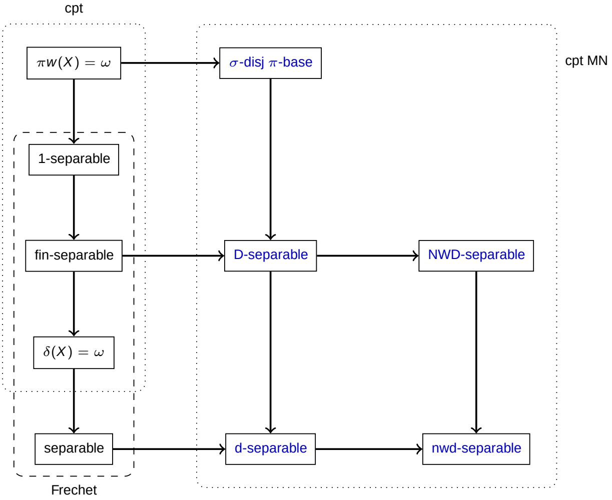
- Then X is not NWD-separable.
- If $E_n \subset D_n$ is nowhere dense, then $E = \bigcup_{n \in \omega} E_n$ is not dense, because it can not contain any $D_n \cap U$.

Positive theorems



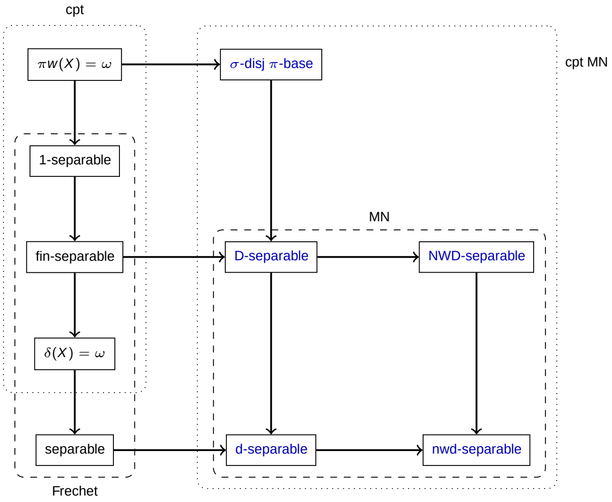
Positive theorems



So-So-Sp.:

- Compact, MN nwd-separable spaces have σ-disjoint π-bases

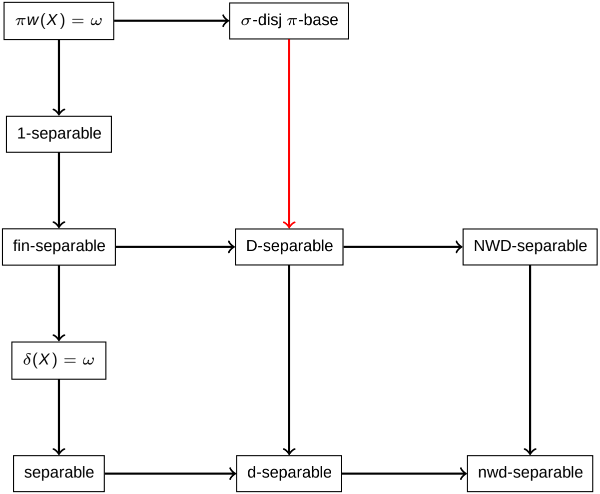
Positive theorems



So-So-Sp.:

- Compact, MN nwd-separable spaces have σ-disjoint π-bases
- MN nwd-separable spaces are D-separable

Countable and compact examples



- Con($\exists$ compact, D-separable, no σ-disjoint π-base)?

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- Thm. If $cof(\mathcal{M}) = \mathfrak{i} = \omega_1$ then there is a countable submaximal space with weight ω_1.

Thank you!