Countable Dense Homogeneous Filters

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Examples:

- the real line is CDH (use back and forth),
- the rationals are NOT CDH (remove one point),
- other spaces known to be CDH: Euclidean spaces, the Cantor set, the Hilbert cube, Hilbert space...

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Question. *For which* 0*-dimensional subsets* X *of* $\mathbb{R}$ *is* ${}^{\omega}X$ *CDH?*

Some Answers

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Notice that by (2) in Theorem 1, it is not possible to extend the result of Theorem 2.

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Notice that in this proof we really found two different countable dense subsets.

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Lemma (Medini and Milovich)**.** *Let $\mathcal{I}$ be an ideal, D a countable dense subset and $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ a homeomorphism. If there is a $x \in \mathcal{I}$ such that $d \triangle h(d) \subset x$ for all $d \in D$, then $h[\mathcal{I}] = \mathcal{I}$.*

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(1) for all $d \in D_0 \cup D_1$, $d \subset^* x$ (intuitively, d finitely misses x),

(2) if $i \in \{0, 1\}$, $d \in D_i$ and we have a partial function $t : n \cap x \to 2$ for $n < \omega$ (that is, some basic open set that is compatible with the x constructed), then there is some $e \in D_{1-i}$ such that $d - x = e - x$ (that is, d and e have the same misses) and e restricted to $n \cap x$ is as t says (so e is in the open set given by t.)

In this way, in the steps of the topological construction we can make $h(d) = e$ for some appropriate t (given by the construction). It is easy to achieve these conditions with an induction, construct such x by finite steps.

But what about $x \in \mathcal{I}$? That's the tricky part.

How we constructed $x \in \mathcal{I}$

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This means that if we do our construction all the ways possible, there will be some one of those that gives $x \in \mathcal{I}$. And we're done!!

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Thank you.