Some results in the extension with a coherent Suslin tree

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Motivation

Theorem (Kunen, Rowbottom, Solovay, etc). $\mathrm{MA}_{\aleph_1}$ *implies* $\mathcal{K}_2$*: Every ccc forcing has property K.*

Question (Todorčević). *Does* $\mathcal{K}_2$ *imply* $\mathrm{MA}_{\aleph_1}$*?*

Theorem (Todorčević). PID $+\mathfrak{p} > \aleph_1$ *implies no* S*-spaces.*

Question (Todorčević). *Under* PID*, does no* S*-spaces imply* $\mathfrak{p} > \aleph_1$*?*

Definition (Todorčević). PFA(S) *is an axiom that there exists a coherent Suslin tree* S *such that the forcing axiom holds for every proper forcing which preserves* S *to be Suslin.*

Theorem (Farah). $\mathfrak{t} = \aleph_1$ *holds in the extension with a Suslin tree.*

Proof. Suppose that T is a Suslin tree, and take $\pi : T \to [\omega]^{\aleph_0}$ such that

$$s \leq_T t \to \pi(s) \supseteq^* \pi(t) \text{ and } s \perp_T t \to \pi(s) \cap \pi(t) \text{ finite.}$$

Then for a generic branch G through T, the set $\{\pi(s) : s \in G\}$ is a $\subseteq^*$-decreasing sequence which doesn't have its lower bound in $[\omega]^{\aleph_0}$. □

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Theorem (Todorčević). $\mathsf{PID} + \mathfrak{p} > \aleph_1$ *implies no* S*-spaces.*

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PFA(S) was introduced to combines many of the consequences of the two contradictory set theoretic axioms, the weak diamond principle, and PFA.

Theorem (Consequences from the weak $\diamondsuit$)**.** *A Suslin tree forces the following.*

- **(Farah)** $\mathfrak{t} = \aleph_1$.
- **(Farah)** *It doesn't hold that all $\aleph_1$-dense subsets of the reals are isomorphic.*
- **(Larson–Todorčević)** *Every ladder system has an ununiformized coloring.*
- **(Larson–Todorčević)** *There are no Q-sets.*
- **(Moore–Hrušák–Džamonja)** $\diamondsuit(\mathbb{R}, \mathbb{R}, \neq)$ *holds.*

Theorem (Consequences from PFA)**.** *Under* PFA(S), S *forces the following.*

- **(Todorčević)** $2^{\aleph_0} = \aleph_2 = \mathfrak{h} = \mathrm{add}(\mathcal{N})$.
- **(Farah)** *The open graph dichotomy.*
- **(Todorčević)** *The P-ideal dichotomy.*
- **(Todorčević)** *There are no compact S-spaces.*

Today, we see the following.

Theorem. *Under* PFA(S), S *forces the following.*

§**1.** *Every forcing with rectangle refining property has precaliber* $\aleph_1$.

§**2.** *There are no* ω_2*-Aronszajn trees.*

§**3.** *All Aronszajn trees are club-isomorphic.*

§**4.** *The weak club guessing and* $\mho$ *fail.*

§1. Every forcing with rec. ref. has precaliber $\aleph_1$ in the ext. with S under PFA(S).

Definition. FSCO_0 *is the collection of forcing notions* $\mathbb{P}$ *such that*

- *conditions of* $\mathbb{P}$ *are finite sets of countable ordinals,*
- $\mathbb{P}$ *is uncountable, and*
- $\leq_{\mathbb{P}}$ *is equal to the superset relation* $\supseteq$*, that is, for any* σ *and* τ *in* $\mathbb{P}$*,* $\sigma \leq_{\mathbb{P}} \tau$ *iff* $\sigma \supseteq \tau$*.*

E.g., a specialization of an Aronszajn tree, freezing an (ω_1, ω_1)-gap, adding an uncountable homogeneous set of a partition.

Definition (Y.)**.** *A forcing notion* $\mathbb{P}$ *in* FSCO_0 *has the rectangle refining property (*REC*) if* $\mathbb{P}$ *is uncountable and*

> *for any* I *and* $J \in [\mathbb{P}]^{\aleph_1}$*, if* $I \cup J$ *forms a* Δ*-system, then there are* $I' \in [I]^{\aleph_1}$ *and* $J' \in [J]^{\aleph_1}$ *s.t. for every* $p \in I'$ *and* $q \in J'$*,* $p \not\perp_{\mathbb{P}} q$*.*

Note that REC implies CCC.

$\mathsf{FSCO}_2 \subseteq \mathsf{FSCO}_0$ is defined (omitted here).

Lemma. *For any ladder system and its colorig, there is a forcing with* REC *in* $FSCO_2$ *which adds a function uniformizing the coloring.*

Theorem (Larson–Todorčević)**.** *In the extension with a coherent Suslin tree, every ladder system has a coloring which cannot be uniformized.*

So, S forces that $MA_{\aleph_1}$(REC in $FSCO_2$) fails.

Lemma. *Under* $MA_{\aleph_1}(S)$*,* S *forces that every forcing with* REC *in* $FSCO_2$ *has precaliber* $\aleph_1$*.*

Therefore,

Theorem. *Under* $MA_{\aleph_1}(S)$*,* S *forces that every forcing with* REC *in* $FSCO_2$ *has precaliber* $\aleph_1$ *and* $MA_{\aleph_1}$(REC in $FSCO_2$) *fails.*

Compare with the following.

Theorem (Todorčević–Veličković)**.** *Every ccc forcing has precaliber* $\aleph_1$ *iff* $MA_{\aleph_1}$ *holds.*

§**2.** There are no ω_2-Aronszajn trees in the extension with S under PFA(S).

This proof is quite standard.

Claim. *For a σ-closed forcing $\mathbb{P}$ and an S-name $\dot{T}$ for an ω_2-tree, $\mathbb{P}$ adds no new S-names for cofinal chains through $\dot{T}$ whenever $\mathfrak{c} > \aleph_1$ holds.*

Claim. *For an S-name $\dot{T}$ for a tree of size $\aleph_1$ and of height ω_1 which doesn't have uncountable (i.e. cofinal) chains through $\dot{T}$, there exists a ccc forcing notion which preserves S to be Suslin and forces $\dot{T}$ to be special (i.e. to be a union of countably many antichains through $\dot{T}$).*

The following is the motivation of this work.

Theorem (Todorčvić)**.** PFA *implies the failure of* $\square_{\kappa,\omega_1}$ *for any unctbl* κ*.*

Theorem (Magidor)**.** *It is consistent that* PFA *and* $\square_{\kappa,\omega_2}$ *hold for any unctbl* κ*.*

Theorem (Magidor)**.** *It is consistent that* PDFA *and* $\square_{\kappa,\omega_1}$ *hold for any unctbl* κ*.*

Theorem (Raghavan)**.** PID *implies the failure of* $\square_{\kappa,\omega}$*, for any unctbl* κ*, and* PID $+\mathfrak{b} > \aleph_1$ *implies the failure of* $\square_{\kappa,\omega_1}$ *for any* κ *with* cf(κ) $> \omega_1$*.*

Question. *Does* PID $+\mathfrak{p} > \aleph_1$ *imply the failure of* $\square_{\omega_1,\omega_1}$*?*

We note that $\square_{\omega_1,\omega_1}$ holds iff there exists a special ω_2-Aronszajn tree.

Claim. *For an S-name $\dot{T}$ for a tree of size $\aleph_1$ and of height ω_1 which doesn't have uncountable (i.e. cofinal) chains through $\dot{T}$, there exists a ccc forcing notion which preserves S to be Suslin and forces $\dot{T}$ to be special (i.e. to be a union of countably many antichains through $\dot{T}$).*

Sketch. Assume that $\dot{<}_T$ is an S-name such that $\Vdash_S$ " $\dot{T} = \langle \omega_1, \dot{<}_T \rangle$ " and for any $s \in S$ and α, β in ω_1, if $s \Vdash_S$ " $\alpha \not<_{\dot{T}} \beta$" and $\alpha < \beta$, then $s \Vdash_S$ " $\alpha \dot{<}_{\dot{T}} \beta$".
Take a club C on ω_1 s.t. for every $\delta \in C$, every node of S_δ decides $\dot{<}_T \cap (\delta \times \delta)$.

$\mathbb{P}$ consists of finite partial functions $p : S \to \bigcup_{\sigma \in [\omega]^{<\aleph_0}} \left([\omega_1]^{<\aleph_0}\right)^\sigma$ such that

- for every $s \in \mathrm{dom}(p)$ and $n \in \mathrm{dom}(p(s))$, $p(s)(n) \subseteq \sup(C \cap \mathrm{lv}(s))$ and
$$s \Vdash_S \text{" } p(s)(n) \text{ is an antichain in } \dot{T}\text{"},$$
- for every s and t in $\mathrm{dom}(p)$, if $s <_S t$, then for every $n \in \mathrm{dom}(p(s)) \cap \mathrm{dom}(p(t))$,
$$t \Vdash_S \text{" } p(s)(n) \cup p(t)(n) \text{ is an antichain in } \dot{T}\text{"},$$

$$p \leq_{\mathbb{P}} q :\Longleftrightarrow p \supseteq q.$$

Note that $\mathbb{P}$ adds an S-name which witnesses that $\dot{T}$ to be special in the extension with S.

It is proved that if $\mathbb{P} \times S$ has an uncountable antichain, then some node of S forces that $\dot{T}$ has an uncountable chain. $\square$

§3. All Aronszajn trees are club-isomorphic in the extension with S under PFA(S).

Let $\dot{T}$ and $\dot{U}$ S-names for Aronszajn trees s.t. $\Vdash_S$ " $\dot{T}, \dot{U} \subseteq \omega^{<\omega_1}$ & $<_{\dot{T}} = <_{\dot{U}} = \subseteq$ ".

$\mathbb{P}$ consists of the functions p such that

- dom(p) is a finite $\in$-chain of countable elementary submodels of $H(\aleph_2)$ with S, $\dot{T}$ and $\dot{U}$,
- for each $M \in$ dom(p), $p(M) = \left\langle t^p_M, f^p_M \right\rangle$, where $t_M \in S$ and $f^p_M : \omega^{\alpha^p_M} \to \omega^{\alpha^p_M}$; non-empty finite partial injection for some $\alpha^p_M <$ ht(t^p_M),
- for each $M, M' \in$ dom(p) with $M' \in M$,

$$t^p_M \notin M,\ t^p_{M'} \in M,\ \alpha^p_M \notin M \text{ and } \alpha^p_{M'} \in M,$$

- for each $M \in \mathrm{dom}(p)$,
  - t^p_M decides the S-names $\dot{T} \cap \omega^{\leq \alpha^p_M}$ and $\dot{U} \cap \omega^{\leq \alpha^p_M}$,
  - $t^p_M \Vdash_S$ " $\mathrm{dom}(f^p_M) \subseteq \dot{T}$ & $\mathrm{ran}(f^p_M) \subseteq \dot{U}$ ", and
  - $t^p_M \Vdash_S$ " $\displaystyle\bigcup_{\substack{M' \in \mathrm{dom}(p) \cap M \\ \text{with } t^p_{M'} <_S t^p_M}} f^p_{M'} \cup f^p_M$ is an order-preserving map whose domain is a subtree of $\dot{T}$ in which every maximal chain is of height $\left|\left\{M' \in \mathrm{dom}(p) \cap M ; t^p_{M'} <_S t^p_M\right\}\right| + 1$ ",

and for each $p = \left\langle \left\langle t^p_M, f^p_M \right\rangle ; M \in \mathrm{dom}(p) \right\rangle$ and $q = \left\langle \left\langle t^q_M, f^q_M \right\rangle ; M \in \mathrm{dom}(q) \right\rangle$ in $\mathbb{P}$,

$$p \leq_{\mathbb{P}} q :\iff \mathrm{dom}(p) \supseteq \mathrm{dom}(q) \ \&\ \forall M \in \mathrm{dom}(q) \left(t^p_M = t^q_M \ \&\ f^p_M \supseteq f^q_M \right).$$

For a $\mathbb{P}$-generic $G_{\mathbb{P}}$, define S-names $\dot{I}_{G_{\mathbb{P}}}$ and $\dot{f}_{G_{\mathbb{P}}}$ such that, letting $\dot{G}_S$ be a canonical S-generic name over the extension by $G_{\mathbb{P}}$,

$$\Vdash_S \text{“ } \dot{I}_{G_{\mathbb{P}}} := \left\{ \alpha^p_M ; p \in G_{\mathbb{P}} \;\&\; M \in \mathsf{dom}(p) \;\&\; t^p_M \in \dot{G}_S \right\} \text{”}$$

and

$$\Vdash_S \text{“ } \dot{f}_{G_{\mathbb{P}}} := \bigcup_{\substack{p \in G_{\mathbb{P}} \\ M \in \mathsf{dom}(p) \\ \text{with } t^p_M \in \dot{G}_S}} f^p_M \text{”}.$$

Note that $\dot{I}_{G_{\mathbb{P}}}$ is an S-name for an uncountable subset of ω_1 and $\dot{f}_{G_{\mathbb{P}}}$ is an $\dot{S}$-name for an isomorphism $\left\{ x \in \dot{T}; \mathsf{ht}(x) \in \dot{I}_{G_{\mathbb{P}}} \right\} \to \left\{ y \in \dot{U}; \mathsf{ht}(y) \in \dot{I}_{G_{\mathbb{P}}} \right\}$.

It is proved that $\mathbb{P}$ is proper and preserves S to be Suslin.

§4. The weak club guessing and $\mho$ fail in the extension with S under PFA(S).

Definition (Shelah). *A ladder system* $\langle C_\alpha; \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ *is called weak club guessing if for every club* $D \subseteq \omega_1$, *there exists* $\alpha \in \omega_1 \cap \mathrm{Lim}$ *such that* $C_\alpha \cap D$ *is unbounded in* α.

Theorem (Shelah ?). PFA *implies no weak club guessing ladder systems.*

Proof. Let $\langle C_\alpha; \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ be a ladder system.
$\mathbb{P}_{\langle C_\alpha;\alpha\in\omega_1\cap\mathrm{Lim}\rangle}$ consists of pairs $p = \langle p_0, p_1\rangle$ such that

- $p_0 : \omega_1 \to \omega_1$; finite partial, strict increasing,
- $p_1 : \omega_1 \cap \mathrm{Lim} \to \omega_1$; finite partial, regressive, and
- for each $\xi \in \mathrm{dom}(p_1)$, $\mathrm{ran}(p_0) \cap C_\xi \subseteq p_1(\xi)$,

$$p \leq_{\mathbb{P}_{\langle C_\alpha;\alpha\in\omega_1\cap\mathrm{Lim}\rangle}} p' :\iff p_0 \supseteq p'_0 \text{ and } p_1 \supseteq p'_1.$$

It suffices to show that $\mathbb{P}_{\langle C_\alpha;\alpha\in\omega_1\cap\mathrm{Lim}\rangle}$ is proper.

$p \in \mathbb{P}_{\langle C_\alpha;\alpha\in\omega_1\cap\mathsf{Lim}\rangle}$ iff
$= \mathbb{P}$

- $p_0 : \omega_1 \to \omega_1$; finite partial, strict increasing,
- $p_1 : \omega_1 \cap \mathsf{Lim} \to \omega_1$; finite partial, regressive, and
- for each $\xi \in \mathrm{dom}(p_1)$, $\mathrm{ran}(p_0) \cap C_\xi \subseteq p_1(\xi)$,

$$p \leq_{\mathbb{P}_{\langle C_\alpha;\alpha\in\omega_1\cap\mathsf{Lim}\rangle}} p' :\Longleftrightarrow\ p_0 \supseteq p'_0 \text{ and } p_1 \supseteq p'_1.$$

Let $\lambda \ll \theta$ be large enough regular, $N \prec \Big\langle H(\theta), \in,$ a Skolem function of $H(\lambda)\Big\rangle$ countable with $\langle C_\alpha; \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$, and $p = \langle p_0, p_1\rangle \in \mathbb{P} \cap N$.

Show that $p^+ = \Big\langle p_0 \cup \Big\{ \langle \omega_1 \cap N, \omega_1 \cap N\rangle \Big\}, p_1\Big\rangle$ is $(N, \mathbb{P})$-generic.

Let $\mathcal{D} \in N$ be dense $\subseteq \mathbb{P}$, and $q \leq_{\mathbb{P}} p^+$ with $q \in \mathcal{D}$. Note that $q_0 \restriction N = q_0 \cap N$.

Take a countable $M \prec H(\lambda)$ in N with $\langle C_\alpha; \alpha \in \omega_1 \cap \mathsf{Lim}\rangle$, $\mathcal{D}$ and $q \cap N$. Then

$$\left\{ r \in \mathcal{D}; r \leq_{\mathbb{P}} q \cap N \text{ and } \Big\{\Big\langle C_\xi \cap M, r_1(\xi)\Big\rangle; \xi \in \mathrm{dom}(r_1) \setminus \mathrm{dom}(q_1 \cap N)\Big\} \supseteq \Big\{\Big\langle C_\xi \cap M, q_1(\xi)\Big\rangle; \xi \in \mathrm{dom}(q_1) \setminus N \text{ with } q_1(\xi) \in N\Big\} \right\}$$

is in M and not empty. Any member r of this set in N is compatible with q.

In fact, for any $r' \leq_{\mathbb{P}_{\langle C_\alpha;\alpha\in\omega_1\cap\mathsf{Lim}\rangle}} r$ in N, r' and $\langle r_0 \cup q_0, r_1 \cup q_1\rangle$ are compatible. $\Box$

Theorem. *Under* PFA(S), S *forces no weak club guessing ladder systems.*

Proof. Let $\left\langle \dot{C}_\alpha : \alpha \in \omega_1 \right\rangle$ be an S-name for a ladder system.
Take a club $E \subseteq \omega_1$ s.t. $\forall \delta \in E$, any nodes of S_δ decides the value of $\dot{C}_\gamma$, $\forall \gamma < \delta$.

$\mathbb{P}_{\left\langle \dot{C}_\alpha : \alpha \in \omega_1 \right\rangle, E}$ consists of finite partial functions p with $\mathrm{dom}(p) \subseteq S$ such that for any $s \in \mathrm{dom}(p)$, $p(s) = \left\langle p_0^s, p_1^s \right\rangle$ such that

- $p_0^s : \sup(E \cap \mathrm{lv}(s)) \to \sup(E \cap \mathrm{lv}(s))$; finite partial, strictly increasing,
- $p_1^s : \omega_1 \to \omega_1$; finite partial, regressive,
- $s \Vdash_S$ " $\left\langle \bigcup_{\substack{t \in \mathrm{dom}(p) \\ \text{with } t \leq_S s}} \mathrm{dom}(p_0^t), \bigcup_{\substack{t \in \mathrm{dom}(p) \\ \text{with } t \leq_S s}} \mathrm{dom}(p_1^t) \right\rangle \in \mathbb{P}_{\langle \dot{C}_\alpha ; \alpha \in \omega_1 \cap \mathrm{Lim} \rangle}$ ",

$$p \leq_{\mathbb{P}_{\left\langle \dot{C}_\alpha : \alpha \in \omega_1 \right\rangle, E}} p' :\iff p \supseteq p'.$$

$\mathbb{P}_{\left\langle \dot{C}_\alpha : \alpha \in \omega_1 \right\rangle, E}$ is proper and preserves S.

In fact,

for any $N \prec \left\langle H(\theta), \in, \text{ a Skolem function of } H(\lambda) \right\rangle$ with S, $\left\langle \dot{C}_\alpha : \alpha \in \omega_1 \right\rangle$ and E,
$p \in \mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E} \cap N$, and $q \in \mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E}$ with $q \leq_{\mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E}} p$,
there exists $q' \leq_{\mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E}} q$ such that
for any $r \in \mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E} \cap N$ with $r \leq_{\mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E}} q' \cap N$,
q' and r are compatible with $\mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E}$.

Therefore every condition of $\mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E} \cap N$ is $(N, \mathbb{P}_{\langle \dot{C}_\alpha : \alpha \in \omega_1 \rangle, E})$-generic.

Compare with the following.

Theorem (Shelah, Moore). *An ω-proper forcing preserves weak club guessing sequences on ω_1.*

$\mho$ case is similar to this.

Recall. A coherent Suslin tree S consists of functions in $\omega^{<\omega_1}$ and closed under finite modifications. That is,

- for any s and t in S, $s \leq_S t$ iff $s \subseteq t$,
- S is closed under taking initial segments,
- for any s and t in S, $\{\alpha \in \min\{\mathsf{lv}(s), \mathsf{lv}(t)\}; s(\alpha) \neq t(\alpha)\}$ is finite, and
- for any $s \in S$ and $t \in \omega^{\mathsf{lv}(s)}$, if $\{\alpha \in \mathsf{lv}(s); s(\alpha) \neq t(\alpha)\}$ is finite, then $t \in S$.

For s and $t \in S$ with the same level, define

$$\begin{array}{cccc} \psi_{s,t} & \{u \in S; s \leq_S u\} & \to & \{u \in S; t \leq_S u\} \\ & \Cup & & \Cup \\ & u & \mapsto & t \cup (u \restriction [\mathsf{lv}(s), \mathsf{lv}(u))) \end{array}.$$

Note that $\psi_{s,t}$ is an isomorphism, and if s, t, u are nodes in S with the same level, then $\psi_{s,t}$, $\psi_{t,u}$ and $\psi_{s,u}$ commute.