Coherent ultrafilters on ccc Boolean algebras

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Outline

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- the lattice of partitions

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- the structure induced by partitions

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- The relation $P \preceq Q$ is a partial order on $Part(\mathbb{B})$.
- $P \wedge Q$ is the infimum of $\{P, Q\}$ in $(Part(\mathbb{B}), \preceq)$.
- $(Part(\mathbb{B}), \wedge, \{1_{\mathbb{B}}\}, \preceq)$ is a semilattice with unit.

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Then $P \vee Q = \{u_p; p \in P\}$ is a partition refined by both P, Q and it is the finest among such partitions. Completeness of $(Part(\mathbb{B}), \preceq)$ implies the existence of a smallest element which necessarily needs to be a partition consisting entirely of atoms of $\mathbb{B}$. In the other direction, the algebra is competely distributive.

Structure induced by partitions

Let $\mathbb{B}$ be a complete ccc Boolean algebra. For $P \in Part(\mathbb{B})$, let $\mathbb{B}_P$ be the subalgebra completely generated by $P \subseteq \mathbb{B}$. Denote the inclusion as $e_P : \mathbb{B}_P \subseteq \mathbb{B}$.

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- The inclusions $e_P : \mathbb{B}_P \subseteq \mathbb{B}$ are regular embeddings.

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Definition. Let $(D, \leq)$ be a directed poset. A set $\{X_\alpha; \alpha \in D\}$ of objects, together with a set $\{f_{\alpha\beta} : X_\alpha \to X_\beta; \alpha \leq \beta \in D\}$ of morphisms, forms a *directed system* if

- $f_{\alpha\alpha} : X_\alpha \to X_\alpha$ is the identity for each $\alpha \in D$
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The algebra as a limit

Fact: Every directed system $\left\{X_\alpha, f_\alpha^\beta; \alpha \leq \beta \in D\right\}$ in the category of Boolean algebras *has* a direct limit that consists of the object $X = (\bigsqcup X_\alpha / \approx)$ where $x \approx y \equiv (\exists \alpha, \beta \leq \gamma \in D)(f_\alpha^\gamma(x) = f_\beta^\gamma(y))$, and morphisms $f_\alpha : X_\alpha \to X$ that send $x \in X_\alpha$ to the equivalence class $[x]_\approx$ of the copy of x in X.

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Theorem: The algebra $\mathbb{B}$, together with the regular embeddings $e_P : \mathbb{B}_P \to \mathbb{B}$, is a direct limit of the directed system $\{\mathbb{B}_P, e_P^Q\}$.

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Proof. Every triangle commutes, i.e. $e_P \circ e_P^Q = e_Q$ whenever $P \preceq Q$. The algebra $\mathbb{B}$ is easily seen to be isomorphic to the limit $(\bigsqcup \mathbb{B}_P / \approx)$ as described above: put $\varphi(x) = [x]_\approx$ for $x \in \mathbb{B}$. The equivalence relation reduces to $x = y$ in $\mathbb{B}$, and merely factorizes out the formal distinction between multiple copies of $x \in \mathbb{B}$ coming from different components $\mathbb{B}_P$ of the disjoint union; hence φ is one-to-one. Clearly, φ is onto and homomorphic.

Ideals and quotients induced by partitions

For $P \in Part(\mathbb{B})$, let $\mathcal{J}_P$ be the ideal on $\mathbb{B}$ generated by $P \subseteq \mathbb{B}$.

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Theorem: The algebra $\mathbb{B}$, together with the quotient mappings $\mathbb{B} \to \mathbb{B}/P$, is an inverse limit of the inverse system of $\mathbb{B}/P$.

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(2) Every infinite complete ccc Boolean algebra is an inverse limit of an inverse system of copies of $P(\omega)/fin$. Dually, every infinite ccc EDC space is a direct limit of directed system of copies of ω^*.

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It is nontrivial if $\mathcal{U} \cap P = \emptyset$.

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The converse is true: every such system determines an ultrafilter $\mathcal{U} = \bigcup \mathcal{U}_P$ on the direct limit algebra $\mathbb{B} = \bigcup \mathbb{B}_P$.

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Coherent P-ultrafilters

Lemma. Let $\mathbb{B}$ be a complete ccc algebra. An ultrafilter $\mathcal{U}$ on $\mathbb{B}$ is a coherent P-ultrafilter iff for every pair of partitions P and Q of $\mathbb{B}$ such that $P \preceq Q$, either $\mathcal{U} \cap Q \neq \emptyset$, or there is a set $X \subseteq P$ such that $\bigvee X \in \mathcal{U}$ and for every $q \in Q$, the set $\{p \in X; p \wedge q \neq 0\}$ is finite.

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But also, consistently **yes**, in a strong sense.

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Question: Is there a coherent P-ultrafilter on a complete ccc Boolean algebra $\mathbb{B}$ which is bigger than $\mathbf{c}$?

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(This slide absolutely doesn't do justice to the whole story.)

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Theorem: Let $\mathbb{B}$ be a complete ccc algebra. Let $\mathcal{U}$ be a coherent P-ultrafilter on $\mathbb{B}$. Then $\mathcal{U}$ is an untouchable point in $St(\mathbb{B})$.

Proof. Let $R = \{\mathcal{F}_n; n \in \omega\} \subseteq St(\mathbb{B})$ be countable nowhere dense. Choose some $a_0 \in \mathcal{F}_0$ with $-a_0 \in \mathcal{U}$ and put $R_0 = \{\mathcal{F} \in R; a_0 \in \mathcal{F}\} \subseteq R$. Continue inductively: if disjoint $a_i \in \mathbb{B}^+$ for $i < k$ have been found such that a_i separates $\mathcal{F}_i$ from $\mathcal{U}$ and $\bigvee_{i<k} a_i \notin \mathcal{U}$ put $R_i = \{\mathcal{F} \in R; a_i \in \mathcal{F}\}$ and consider $\bigcup_{i<k} R_i \subseteq R$. If $\bigcup_{i<k} R_i = R$, good for $\mathcal{U}$. Otherwise, let n_k be the first index such that $\mathcal{F}_{n_k} \notin \bigcup_{i<k} R_i$ and choose some $a_k \perp \bigvee_{i<k} a_i$ separating $\mathcal{F}_{n_k}$ from $\mathcal{U}$. This either stops at some k or we arrive at an infinite disjoint system $Q = \{a_i; i \in \omega\} \subseteq \mathbb{B}^+$. WLOG, Q is a partition. For each $a_i \in Q$, choose an infinite partition P_i of a_i such that $P_i \cap \bigcup R_i = \emptyset$ (R_i is nowhere dense). So $P = \bigcup P_i \preceq Q$.

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