Selection principles and dense sets

Santi Spadaro

Ben Gurion University of the Negev
(Supported by the Center for Advanced Studies in Mathematics)
santi@bgu.ac.il

Winter School in Abstract Analysis
Hejnice, Czech Republic, 2011

Notation

- (X, τ) always denotes a Hausdorff (T_2) topological space.

Notation

- (X, τ) always denotes a Hausdorff (T_2) topological space.
- A π-base for (X, τ) is a dense set in the POSET $(\tau \setminus \{\emptyset\}, \subseteq)$.

Notation

- (X, τ) always denotes a Hausdorff (T_2) topological space.
- A π-base for (X, τ) is a dense set in the POSET $(\tau \setminus \{\emptyset\}, \subseteq)$.
- A set $E \subset X$ is *discrete* if for every $x \in E$ there is an open $U \subset X$ such that $U \cap E = \{x\}$.

Notation

- (X, τ) always denotes a Hausdorff (T_2) topological space.
- A π-base for (X, τ) is a dense set in the POSET $(\tau \setminus \{\emptyset\}, \subseteq)$.
- A set $E \subset X$ is *discrete* if for every $x \in E$ there is an open $U \subset X$ such that $U \cap E = \{x\}$.
- (Spread of X) $s(X) = \sup\{|D| : D \subset X \text{ is discrete }\}$.

Notation

- (X, τ) always denotes a Hausdorff (T_2) topological space.
- A π-base for (X, τ) is a dense set in the POSET $(\tau \setminus \{\emptyset\}, \subseteq)$.
- A set $E \subset X$ is *discrete* if for every $x \in E$ there is an open $U \subset X$ such that $U \cap E = \{x\}$.
- (Spread of X) $s(X) = \sup\{|D| : D \subset X \text{ is discrete }\}$.
- $C_p(X, 2) = \{f \in 2^X : f \text{ is continuous }\}$, with the relative topology from 2^X.

Notation

- (X, τ) always denotes a Hausdorff (T_2) topological space.
- A π-base for (X, τ) is a dense set in the POSET $(\tau \setminus \{\emptyset\}, \subseteq)$.
- A set $E \subset X$ is *discrete* if for every $x \in E$ there is an open $U \subset X$ such that $U \cap E = \{x\}$.
- (Spread of X) $s(X) = \sup\{|D| : D \subset X \text{ is discrete }\}$.
- $C_p(X, 2) = \{f \in 2^X : f \text{ is continuous }\}$, with the relative topology from 2^X.

Theorem

(Hewitt-Marczewski-Pondiczery Light) 2^κ is separable, for every $\kappa \leq \mathfrak{c}$.

Notation

- (X, τ) always denotes a Hausdorff (T_2) topological space.
- A π-base for (X, τ) is a dense set in the POSET $(\tau \setminus \{\emptyset\}, \subseteq)$.
- A set $E \subset X$ is *discrete* if for every $x \in E$ there is an open $U \subset X$ such that $U \cap E = \{x\}$.
- (Spread of X) $s(X) = \sup\{|D| : D \subset X \text{ is discrete }\}$.
- $C_p(X, 2) = \{f \in 2^X : f \text{ is continuous }\}$, with the relative topology from 2^X.

Theorem

(Hewitt-Marczewski-Pondiczery Light) 2^κ is separable, for every $\kappa \leq \mathfrak{c}$.

Proof.

$C_p(2^\omega, 2)$ is a dense countable subset of $2^{2^\omega} = 2^{\mathfrak{c}}$. □

A selective version of separability

Definition

(Scheepers) A space X is R-separable if for any sequence $\{D_n : n < \omega\}$ of dense sets you can pick points $x_n \in D_n$ such that $\{x_n : n < \omega\}$ is dense.

A selective version of separability

Definition

(Scheepers) A space X is R-separable if for any sequence $\{D_n : n < \omega\}$ of dense sets you can pick points $x_n \in D_n$ such that $\{x_n : n < \omega\}$ is dense.

$\pi w(X) = \aleph_0 \Rightarrow R$-separable $\Rightarrow \delta(X) = \aleph_0$ (i.e., every dense set is separable).

A selective version of separability

Definition

(Scheepers) A space X is R-separable if for any sequence $\{D_n : n < \omega\}$ of dense sets you can pick points $x_n \in D_n$ such that $\{x_n : n < \omega\}$ is dense.

$\pi w(X) = \aleph_0 \Rightarrow R$-separable $\Rightarrow \delta(X) = \aleph_0$ (i.e., every dense set is separable).

Example

A countable non-R-separable space.

Proof.

- $X = Fn(\omega, \omega)$.
- $F \in X$, $\mathcal{F} \in [\omega^\omega]^{<\omega}$, $V(F, \mathcal{F}) := \{G \in X : G \supset F \wedge (\forall f \in \mathcal{F})(\forall n \in domG \setminus domF)(G(n) \neq f(n))\}$
- Declare the Vs to be a local base at F.
- $D_n = \{F \in X : n \in dom(F)\}$ is dense, but no selection clusters to $\emptyset$.

R-separability and Cantor Cubes I

$cov(\mathcal{M})$=minimum number of meager sets needed to cover $\mathbb{R}$.

R-separability and Cantor Cubes I

$cov(\mathcal{M})$=minimum number of meager sets needed to cover $\mathbb{R}$.

Theorem

(Bella, Bonanzinga and Matveev) If $\kappa = \pi w(X) < cov(\mathcal{M})$ and $\delta(X) = \omega$ then X is R-separable.

R-separability and Cantor Cubes I

$cov(\mathcal{M})$=minimum number of meager sets needed to cover $\mathbb{R}$.

Theorem

(Bella, Bonanzinga and Matveev) If $\kappa = \pi w(X) < cov(\mathcal{M})$ and $\delta(X) = \omega$ then X is R-separable.

Proof.

- (Arnie Miller, 1982) $cov(\mathcal{M})$ minimum size of a subfamily of ω^ω which cannot be guessed.

R-separability and Cantor Cubes I

$cov(\mathcal{M})$=minimum number of meager sets needed to cover $\mathbb{R}$.

Theorem

(Bella, Bonanzinga and Matveev) If $\kappa = \pi w(X) < cov(\mathcal{M})$ and $\delta(X) = \omega$ then X is R-separable.

Proof.

- (Arnie Miller, 1982) $cov(\mathcal{M})$ minimum size of a subfamily of ω^ω which cannot be guessed.
- $D_n = \{d_{n,m} : m < \omega\}$ countable dense. $\{B_\alpha : \alpha < \kappa\}$ a π-base.
- $f_\alpha(n) = \min\{m : d_{nm} \in B_\alpha\}$.
- Take f guessing all the f_α's.
- Then $\{x_{n,f(n)} : n < \omega\}$ is dense in X.

□

R-separability and Cantor Cubes II

$cov(\mathcal{M})$ is the least size of a subset of $\mathbb{R}$ which does not have strong measure zero.

R-separability and Cantor Cubes II

$cov(\mathcal{M})$ is the least size of a subset of $\mathbb{R}$ which does not have strong measure zero.

Theorem

(Bella, Bonanzinga and Matveev) $2^{cov(\mathcal{M})}$ *contains a dense countable non-R-separable space.*

R-separability and Cantor Cubes II

$cov(\mathcal{M})$ is the least size of a subset of $\mathbb{R}$ which does not have strong measure zero.

Theorem

(Bella, Bonanzinga and Matveev) $2^{cov(\mathcal{M})}$ contains a dense countable non-R-separable space.

Proof.

1. Work on 2^X, where $X \subset \mathbb{R}$ strong non-measure zero set of minimal size.
2. Not strong measure zero $\iff (\exists\{\epsilon_n : n < \omega\})(X \not\subseteq \{U_n : n < \omega\}$ whenever $\mu(U_n) < \epsilon_n)$.
3. Let $\mathcal{B}_n$ be the set of all traces on X of finite unions of intervals with rational endpoints of measure $< \epsilon_n$.
4. Let $D_n = \{\chi_B : B \in \mathcal{B}_n\}$.
5. each D_n is dense, but $(D_n : n < \omega)$ has no dense selection.

R-separability OF Cantor Cubes

- $\kappa_1 :=$ the least cardinal κ such that 2^κ contains a countable non-R-separable dense subspace.
- $\kappa_1 = cov(\mathcal{M})$

R-separability OF Cantor Cubes

- $\kappa_1 :=$ the least cardinal κ such that 2^κ contains a countable non-R-separable dense subspace.
- $\kappa_1 = cov(\mathcal{M})$
- $\kappa_2 :=$ the least cardinal κ such that 2^κ is not R-separable.

R-separability OF Cantor Cubes

- $\kappa_1 :=$ the least cardinal κ such that 2^κ contains a countable non-R-separable dense subspace.
- $\kappa_1 = cov(\mathcal{M})$
- $\kappa_2 :=$ the least cardinal κ such that 2^κ is not R-separable.
- $\kappa_2 = \omega_1$

R-separability OF Cantor Cubes

- $\kappa_1 :=$ the least cardinal κ such that 2^κ contains a countable non-R-separable dense subspace.
- $\kappa_1 = cov(\mathcal{M})$
- $\kappa_2 :=$ the least cardinal κ such that 2^κ is not R-separable.
- $\kappa_2 = \omega_1$
- because $\sigma(2^{\omega_1}) = \{f \in 2^{\omega_1} : |supp(f)| < \omega\}$ is dense in 2^{ω_1} but not separable.

A more discreet version of R-separability

Definition

A space is D-separable if for every sequence $\{D_n : n < \omega\}$ of dense sets there are discrete sets $E_n \subset D_n$ such that $\bigcup_{n<\omega} E_n$ is dense.

Definition

A space is *d-separable* if it has a σ-discrete dense set.

σ-disjoint π-base $\Rightarrow$ D-separability $\Rightarrow$ d-separability.

General facts about d and D-separability

- d-separability was introduced by Kurepa in his PhD thesis.

General facts about d and D-separability

- d-separability was introduced by Kurepa in his PhD thesis.
- (Arhangel'skii) $\prod_{i \in I} X_i$ is d-separable whenever every X_i is d-separable.
- (Matveev) $X^{2^{d(X)}}$ is never D-separable.

General facts about d and D-separability

- d-separability was introduced by Kurepa in his PhD thesis.
- (Arhangel'skii) $\prod_{i \in I} X_i$ is d-separable whenever every X_i is d-separable.
- (Matveev) $X^{2^{d(X)}}$ is never D-separable.
- (trivial) If $X = \bigcup_{i<n} X_i$ and every X_i is d-separable then X is d-separable.
- (non-trivial) If $X = \bigcup_{i<n} X_i$ and every X_i is D-separable then X is D-separable.

General facts about d and D-separability

- d-separability was introduced by Kurepa in his PhD thesis.
- (Arhangel'skii) $\prod_{i \in I} X_i$ is d-separable whenever every X_i is d-separable.
- (Matveev) $X^{2^{d(X)}}$ is never D-separable.
- (trivial) If $X = \bigcup_{i<n} X_i$ and every X_i is d-separable then X is d-separable.
- (non-trivial) If $X = \bigcup_{i<n} X_i$ and every X_i is D-separable then X is D-separable.
- For every space X there is a space Y such that $X \times Y$ is D-separable.

General facts about d and D-separability

- d-separability was introduced by Kurepa in his PhD thesis.
- (Arhangel'skii) $\prod_{i \in I} X_i$ is d-separable whenever every X_i is d-separable.
- (Matveev) $X^{2^{d(X)}}$ is never D-separable.
- (trivial) If $X = \bigcup_{i<n} X_i$ and every X_i is d-separable then X is d-separable.
- (non-trivial) If $X = \bigcup_{i<n} X_i$ and every X_i is D-separable then X is D-separable.
- For every space X there is a space Y such that $X \times Y$ is D-separable.
- There are countable non-D-separable spaces.

"Forcing" a countable non D-separable space (Soukup)

Let $\mathcal{D}$ be a colllection of dense sets in X.

Definition

A $\mathcal{D}$-mosaic is a set of the form $\bigcup_{U \in \mathcal{U}} U \cap D_U$, where $D_U \in \mathcal{D}$ and $\mathcal{U}$ is a maximal pairwise disjoint family of open sets.

Definition

A space is $\mathcal{D}$-forced if every dense set contains a $\mathcal{D}$-mosaic.

Theorem

(Juhász, Soukup and Szentmiklóssy) There are countable $\mathcal{D}$-forced dense subspaces $2^{\mathfrak{c}}$

- $\mathfrak{cDs} = \min\{\kappa : 2^\kappa$ contains a countable non-D-separable subspace$\}$.

- $\mathfrak{cds} = \min\{\kappa : 2^\kappa$ contains a countable non-D-separable subspace$\}$.
- $\omega_1 \leq \mathfrak{cds} \leq \mathfrak{c}$.
- What is $\mathfrak{cds}$?

- $\mathfrak{cds} = \min\{\kappa : 2^\kappa$ contains a countable non-D-separable subspace$\}$.
- $\omega_1 \leq \mathfrak{cds} \leq \mathfrak{c}$.
- What is $\mathfrak{cds}$?
- $\mathfrak{ds} = \min\{\kappa : 2^\kappa$ is not D-separable $\}$.

- $\mathfrak{cds} = \min\{\kappa : 2^\kappa$ contains a countable non-D-separable subspace$\}$.
- $\omega_1 \leq \mathfrak{cds} \leq \mathfrak{c}$.
- What is $\mathfrak{cds}$?
- $\mathfrak{ds} = \min\{\kappa : 2^\kappa$ is not D-separable $\}$.

Theorem

$\mathfrak{ds} \leq \omega_2$.

- $\mathfrak{cds} = \min\{\kappa : 2^\kappa$ contains a countable non-D-separable subspace$\}$.
- $\omega_1 \leq \mathfrak{cds} \leq \mathfrak{c}$.
- What is $\mathfrak{cds}$?
- $\mathfrak{ds} = \min\{\kappa : 2^\kappa$ is not D-separable $\}$.

Theorem

$\mathfrak{ds} \leq \omega_2$.

Proof.

1. (Juhász? Szentmiklóssy?) There is a Strong $\aleph_2 - HFD_w$.
2. $(\exists X)(\forall n)(s(X^n) = \aleph_1 \wedge d(C_p(X,2)) = |X| = \aleph_2)$.
3. Assume $C_p(X,2)$ is d-separable. Then $s(C_p(X,2)) = \aleph_2$.
4. $s(C_p(X,2)) = \aleph_2 \rightarrow (\exists n)(s(X^n) = \aleph_2)$.

□

Products I

Theorem

(CH) There are countable R-separable spaces X and Y such that $X \times Y$ is not D-separable.

Theorem

Let X be any space and Y be a space such that for some $j < \omega$, $\hat{c}(Y^j) \geq \pi w(Y) \geq \pi w(X)$. Then $X \times Y^\mu$ has a σ-disjoint π-base for every $\mu \in [\omega, \kappa]$.

Corollary

For every X there is Y such that $X \times Y$ has a σ-disjoint π-base.

Proof.

If $\kappa = \pi w(X)$, simply choose $Y = D(\kappa)^\omega$. □

Products II

Theorem

Let X be any space and Y be a space such that for some $j < \omega$, $\hat{c}(Y^j) \geq \pi w(Y) \geq \pi w(X)$. Then $X \times Y^\mu$ has a σ-disjoint π-base for every $\mu \in [\omega, \kappa]$.

Corollary

If X is a LOTS, then X^μ has a σ-disjoint π-base, for every $\mu \in [\omega, d(X)]$.

Proof.

A result of Petr Simon from 1973 says that X^2 has a cellular family of size $d(X) = \pi w(X)$. Now $(X^2)^\mu = X^\mu$ for μ infinite. □

Thank you!

A. Bella, M. Matveev and S. Spadaro, *Variations of selective separability II: discrete sets and the influence of convergence and maximality*, submitted (http://arxiv.org/abs/1101.4615).