On monotone hull operations

Marek Balcerzak, Tomasz Filipczak

Institute of Mathematics, Technical University of Łódź, ul. Wólczańska 215,
93-005 Łódź, Poland

Definitions and notation

Fix a triple $(X, \mathcal{S}, \mathcal{I})$ where $X \neq \emptyset$, $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Such a triple will be called a **measurable space with negligibles**.

For $A, B \subseteq X$ we write $A \subseteq_{\mathcal{I}} B$ whenever $A \setminus B \in \mathcal{I}$.
Symbol $\triangle$ denotes symmetric difference of sets.

We say that $H \in \mathcal{S}$ is a **hull** (with respect to $(X, \mathcal{S}, \mathcal{I})$) of a set $A \subseteq X$ if H contains A and for every $G \in \mathcal{S}$ containing A, we have $H \subseteq_{\mathcal{I}} G$.

If additionally, $H \in \mathcal{H}$ (for a given family $\mathcal{H} \subseteq \mathcal{S}$), we say that H is an $\mathcal{H}$-**hull** of A. Observe that every hull of $A \in \mathcal{I}$ is in $\mathcal{I}$.

If $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$, we say that $\varphi \colon \mathcal{A} \to \mathcal{H}$ is an $\mathcal{H}$-**hull operation** on $\mathcal{A}$ whenever $\varphi(A)$ is an $\mathcal{H}$-hull of A for each $A \in \mathcal{A}$. If $\varphi(A) \subseteq \varphi(B)$ for any $A, B \in \mathcal{A}$ with $A \subseteq B$, then φ is called **monotone**.

Our aim is to generalize some results of **Elekes and Máthé (2009)** on monotone Borel hull operations with respect to $(\mathbb{R}, \mathcal{L}, \mathcal{N})$ where $\mathcal{L}$ stands for the σ-algebra of Lebesgue measurable sets and $\mathcal{N}$ denotes the σ-ideal of Lebesgue null sets.

They proved in particular that the existence of a monotone $\mathrm{Borel}(\mathbb{R})$-hull operation on $\mathcal{P}(\mathbb{R})$ (or on $\mathcal{L}$, or on $\mathcal{N}$) is independent of ZFC.

We will extend their ideas to the category case (dealing with $(\mathbb{R}, \mathcal{K}, \mathcal{M})$ where $\mathcal{K}$ stands for the σ-algebra of Baire sets and $\mathcal{M}$ denotes the σ-ideal of meager sets.

We will also work with two measurable spaces with negligibles associated with the product σ-ideals $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ on $\mathbb{R}^2$.

Recall some cardinals associated with ideals. Fix an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$ with $\bigcup \mathcal{I} \notin \mathcal{I}$. Define:

- $\mathrm{non}(\mathcal{I}) := \min\{|A|\colon A \subseteq X,\ A \notin \mathcal{I}\}$;
- $\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{F}|\colon \mathcal{F} \subseteq \mathcal{I},\ \bigcup \mathcal{F} \notin \mathcal{I}\}$;
- $\mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{F}|\colon \mathcal{F} \subseteq \mathcal{I},\ (\forall A \in \mathcal{I})(\exists B \in \mathcal{F}) A \subseteq B\}$.

Note that add$(\mathcal{I}) \leq$ cof$(\mathcal{I})$. Every family $\mathcal{D} \subseteq \mathcal{I}$ such that

$$(\forall A \in \mathcal{I})(\exists B \in \mathcal{D}) A \subseteq B$$

will be called **cofinal** in $\mathcal{I}$, or a **base** of $\mathcal{I}$. Clearly, every base $\mathcal{D}$ of $\mathcal{I}$ generates a $\mathcal{D}$-hull operation $\varphi\colon \mathcal{I} \to \mathcal{D}$.

If $\mathcal{E} \subseteq \mathcal{P}(X)$ we denote by $\mathcal{E} \triangle \mathcal{I}$ the family of all sets of the form $E \triangle A$ where $E \in \mathcal{E}$ and $A \in \mathcal{I}$.

If $\mathcal{E}$ is a σ-algebra and $\mathcal{I}$ is a σ-ideal then $\mathcal{E} \triangle \mathcal{I}$ is the smallest σ-algebra containing $\mathcal{E} \cup \mathcal{I}$.

For $\mathcal{H} \subseteq \mathcal{P}(X)$, we denote by $\mathcal{H}_c$, $\mathcal{H}_\sigma$, $\mathcal{H}_\delta$, respectively, the families consisting of all complements, countable unions, and countable intersections of sets from $\mathcal{H}$. We can use these operations more than once, so we consider families of type $\mathcal{H}_{\sigma\delta\sigma}$, etc.

In the sequel, the σ-algebra of Borel sets in $\mathbb{R}^n$ will be denoted by $\mathcal{B}(\mathbb{R}^n)$ or briefly by $\mathcal{B}$, if it is clear in which space we work.

We denote by $\mathcal{F}_\sigma \sqcup \mathcal{G}_\delta$ the family of all subsets of a given space that can be expressed in the form $A \cup B$ where A is of type F_σ and B is of type G_δ.

Now, we will recall some facts concerning $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$. Assume that $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ are σ-ideals. For $A \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}$ we put

$$A(x) := \{y \in \mathbb{R} \colon (x, y) \in A\}, \;\; x\text{-section of } A,$$

$$\pi_{\mathcal{J}}(A) := \{x \in \mathbb{R} \colon A(x) \notin \mathcal{J}\}, \;\; \mathcal{J}\text{-projection of } A.$$

Then define a family

$$\mathcal{I} \otimes \mathcal{J} := \{A \subseteq \mathbb{R}^2 \colon (\exists B \in \mathcal{B}(\mathbb{R}^2))(A \subseteq B \text{ and } \pi_{\mathcal{J}}(B) \in \mathcal{I})\}$$

which forms a σ-ideal called the **Fubini product** of $\mathcal{I}$ and $\mathcal{J}$. Note that $\mathcal{I} \otimes \mathcal{J}$, by the definition, possesses a base of Borel sets.

In particular, $\mathcal{N}\otimes\mathcal{N}$ and $\mathcal{M}\otimes\mathcal{M}$ coincide with the σ-ideals of Lebesgue null sets and of meager sets in $\mathbb{R}^2$.

The mixed product σ-ideals $\mathcal{M}\otimes\mathcal{N}$ and $\mathcal{N}\otimes\mathcal{M}$ are called the **Mendez σ-ideals**; see [**Mendez, 1976**].

Note that these σ-ideals are mutually incomparable (with respect to inclusion) and also incomparable with $\mathcal{M}\otimes\mathcal{M}$ and $\mathcal{N}\otimes\mathcal{N}$.

Fact 1 (folklore).

$$\mathrm{non}(\mathcal{M}\otimes\mathcal{N}) = \max\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\} = \mathrm{non}(\mathcal{N}\otimes\mathcal{M}).$$

Fact 2. *For each of the triples*

$$(\mathbb{R}, \mathcal{K}, \mathcal{M}), \quad (\mathbb{R}^2, \mathcal{B} \triangle (\mathcal{N} \otimes \mathcal{M}), \mathcal{N} \otimes \mathcal{M}), \quad (\mathbb{R}^2, \mathcal{B} \triangle (\mathcal{M} \otimes \mathcal{N}), \mathcal{M} \otimes \mathcal{N})$$

there exists a monotone $\mathcal{S}$-hull operation on $\mathcal{P}(\mathbb{R})$ or $\mathcal{P}(\mathbb{R}^2)$, respectively, where $\mathcal{S}$ is the respective σ-algebra.

The same holds for $(\mathbb{R}, \mathcal{L}, \mathcal{N})$ which was proved by Elekes and Máthé.

In all those cases a role of $\mathcal{S}$-hull can be played by the operator of closure $A \longmapsto \overline{A}$ for the respective density-type topology.

Results

Question: Is there a **monotone Borel hull** (on a given σ-ideal, σ-algebra, or on $\mathcal{P}(\mathbb{R}^n)$)?

Our results will be proved with the use of methods analogous to those used by Elekes and Máthé.

First we deal with monotone Borel hull operations on σ-ideals. Recall the negative result by Elekes and Máthé.

Theorem 3. *In a model obtained by adding ω_2 Cohen reals to a model satisfying the Continuum Hypothesis (CH) there is no monotone Borel hull operation on $\mathcal{N}$.*

For the category we have the following analogous theorem.

Theorem 4. *In a model obtained by adding ω_2 random reals to a model satisfying CH there is no monotone Borel hull operation on $\mathcal{M}$.*

Proof. In the considered model (it is due to Kunen) we have $\mathrm{non}(\mathcal{M}) = \omega_2 = 2^\omega$. Also in this model there is no strictly increasing (with respect to inclusion) sequence of Borel subsets of the reals which is well ordered in type ω_2.

Then suppose that $\varphi\colon \mathcal{M} \to \mathcal{B}$ is a monotone hull operation. Pick $H := \{x_\alpha\colon \alpha < \mathrm{non}(\mathcal{M})\} \notin \mathcal{M}$ and consider $B_\alpha := \varphi(\{x_\beta\colon \beta < \alpha\})$ for $\alpha < \mathrm{non}(\mathcal{M})$. Observe that the sequence $\{B_\alpha\colon \alpha < \mathrm{non}(\mathcal{M})\}$ of Borel sets cannot stabilise since in this case H would be contained in a meager set. But then we can select a strictly increasing subsequence of length ω_2, a contradiction. ∎

Using Fact 1 and an argument analogous to the final part of the proof of Theorem 4 we obtain

Corollary 5. *Let $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$. Consider a model obtained by adding either ω_2 Cohen reals or ω_2 random reals to a model satisfying CH. In this model there is no monotone Borel hull operation on $\mathcal{I}$.*

On the other hand, we have the following positive result whose proof goes analogously as in the measure case.

Proposition 6. *Assume that $\mathcal{I}$ is a σ-ideal with $\mathrm{cof}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$ and let $\mathcal{H}$ be a fixed base of $\mathcal{I}$. Then there exists a monotone $\mathcal{H}$-hull operation on $\mathcal{I}$.*

Proof. Let $\{A_\alpha \colon \alpha < \mathrm{cof}(\mathcal{I})\}$ be cofinal in $\mathcal{I}$. By recursion, for every $\alpha < \mathrm{cof}(\mathcal{I})$ pick $B_\alpha \in \mathcal{H}$ such that $A_\alpha \cup \bigcup_{\beta<\alpha} B_\beta \subseteq B_\alpha$. Then $\{B_\alpha \colon \alpha < \mathrm{cof}(\mathcal{I})\}$ is a cofinal increasing sequence of sets in $\mathcal{H}$. For each $I \in \mathcal{I}$ define $\varphi(I)$ as B_{α_I} where $\alpha_I < \mathrm{cof}(\mathcal{I})$ is the minimal index with $I \subseteq B_{\alpha_I}$. Then $\varphi \colon \mathcal{I} \to \mathcal{H}$ satisfies the assertion. ∎

If a σ-ideal $\mathcal{I}$ has a base consisting of Borel sets then $\mathrm{cof}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$ follows from CH. We have the following corollaries.

Corollary 7. *Assume* $\mathrm{cof}(\mathcal{M}) = \mathrm{add}(\mathcal{M})$. *Then there exists a monotone* $\mathcal{F}_\sigma$-*hull operation on* $\mathcal{M}$.

Corollary 8. *Let* $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$ *and assume that* $\mathrm{cof}(\mathcal{I}) = \mathrm{add}(\mathcal{I})$. *Then there exists a monotone* $\mathcal{F}_\sigma \sqcup \mathcal{G}_\delta$-*hull operation on* $\mathcal{I}$.

Note that

$$\mathrm{cof}(\mathcal{N} \otimes \mathcal{M}) = \mathrm{cof}(\mathcal{N}), \quad \mathrm{add}(\mathcal{N} \otimes \mathcal{M}) = \mathrm{add}(\mathcal{N}),$$
$$\mathrm{cof}(\mathcal{M} \otimes \mathcal{N}) = \mathrm{cof}([\mathbb{R}]^{\leq \omega}),$$
$$\mathrm{add}(\mathcal{M} \otimes \mathcal{N}) = \mathrm{add}([\mathbb{R}]^{\leq \omega}) = \omega_1.$$

This is due to **Cichoń and Pawlikowski, 1986**.

Monotone Borel hull operations on σ-algebras

Now, we consider a measurable space with negligibles $(X, \mathcal{S}, \mathcal{I})$. We ask about the existence of monotone $\mathcal{H}$-hull operations on $\mathcal{P}(X)$ and $\mathcal{S}$ for some good subfamilies $\mathcal{H}$ of $\mathcal{S}$.

Observe that if there exist a monotone $\mathcal{S}$-hull operation on $\mathcal{P}(X)$ and a monotone $\mathcal{H}$-hull operation on $\mathcal{S}$, then their composition is a monotone $\mathcal{H}$-hull operation on $\mathcal{P}(X)$.

Hence by Fact 2 it follows that, for triples

$$(\mathbb{R}, \mathcal{K}, \mathcal{M}), \quad \left(\mathbb{R}^2, \mathcal{B} \triangle (\mathcal{N} \otimes \mathcal{M}), \mathcal{N} \otimes \mathcal{M}\right), \quad \left(\mathbb{R}^2, \mathcal{B} \triangle (\mathcal{M} \otimes \mathcal{N}), \mathcal{M} \otimes \mathcal{N}\right),$$

the existence of a monotone $\mathcal{H}$-hull operation on $\mathcal{P}(\mathbb{R})$ (or $\mathcal{P}\left(\mathbb{R}^2\right)$) is equivalent to the existence of a monotone $\mathcal{H}$-hull operation on $\mathcal{K}$ (or $\mathcal{B} \triangle \mathcal{I}$ with $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$). For $(\mathbb{R}, \mathcal{L}, \mathcal{N})$ this was observed by Elekes and Máthé.

Theorem 9. *Let* $\mathcal{H} \subseteq \mathcal{P}(X)$ *be a finitely additive and countably multiplicative family with* $|\mathcal{H}| \leq \omega_1$, $\mathcal{H}_c \subseteq \mathcal{H}_\sigma$, *and let* $\mathcal{I} \subseteq \mathcal{P}(X)$ *be a* σ*-ideal such that* $\mathcal{S} := \mathcal{H} \triangle \mathcal{I}$ *forms a* σ*-algebra. If* $\mathcal{I}$ *has a base of sets in* $\mathcal{H}$*, then there exists a monotone* $\mathcal{H}_{c\delta\sigma}$*-hull operation on* $\mathcal{S}$.

Corollary 10. *Let* $\mathcal{H} \subseteq \mathcal{P}(X)$ *be a* σ*-algebra with* $|\mathcal{H}| \leq \omega_1$, *and let* $\mathcal{I} \subseteq \mathcal{P}(X)$ *be a* σ*-ideal. If* $\mathcal{I}$ *has a base of sets in* $\mathcal{H}$*, then there exists a monotone* $\mathcal{H}$*-hull operation on* $\mathcal{H} \triangle \mathcal{I}$.

This corollary applied, **under CH**, with $\mathcal{H} := \mathcal{B}$ to $\mathcal{N}$, $\mathcal{M}$ and to the Mendez ideals, yields the respective monotone $\mathcal{B}$-hull operations. However, we want to obtain these operations with values in possibly low Borel classes.

Theorem 9 can be applied to $\mathcal{N} \otimes \mathcal{M}$ and $\mathcal{M} \otimes \mathcal{N}$ with $\mathcal{H} := F_{\sigma\delta}$. Hence we have

Theorem 11 (CH). *If $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$, then there exists a monotone $\mathcal{G}_{\delta\sigma\delta\sigma}$-hull operation on $\mathcal{B} \triangle \mathcal{I}$ (on $\mathcal{P}(\mathbb{R}^2)$).*

Note that Theorem 9 applied to $\mathcal{I} := \mathcal{N}$ and $\mathcal{H} := \mathcal{G}_\delta$ yields a monotone $\mathcal{F}_{\sigma\delta\sigma}$-hull on $\mathcal{L}$, under CH [**Elekes and Máthé**].

In the category case we have

Theorem 12 (CH). *There exists a monotone $\mathcal{G}_{\delta\sigma}$-hull operation on $\mathcal{K}$ (on $\mathcal{P}(\mathbb{R})$).*

References

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2. **M. Balcerzak, T. Filipczak**, *On monotone hull operations*, Math. Log. Quart, (2011), accepted.