Subcompact cardinals, squares and stationary reflection

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Squares

Recall Jensen's $\square$ principle:

Definition

For any cardinal α, a $\square_\alpha$-sequence is a sequence $\langle C_\beta \mid \beta \in \alpha^+ \cap Lim\rangle$ such that for every $\beta \in \alpha^+ \cap Lim$,

- *C_β is a closed unbounded subset of β,*
- *$ot(C_\beta) \leq \alpha$,*
- *for any $\gamma \in \lim(C_\beta)$, $C_\gamma = C_\beta \cap \gamma$.*

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$\square_\alpha$ is really more a property of α^+ than α.

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Questions:

Can we show that subcompactness is really optimal? What about cardinals other than the one with the large cardinal property?

Generalising Jensen's subcompactness

Recall that for any cardinal α, we denote by H_α the set of all sets whose transitive closure has cardinality strictly less than α.

Definition

For any cardinal α, we say that a cardinal $\kappa < \alpha$ is *α-subcompact* if for every $A \subseteq H_\alpha$, there exist $\bar{\alpha} < \alpha$ and $\bar{A} \subseteq H_{\bar{\alpha}}$ such that there is an elementary embedding

$$\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \to (H_\alpha, \in, A)$$

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In this terminology, Jensen's original notion of subcompactness is κ^+-subcompactness. Also note that if $\kappa < \beta < \alpha$ and κ is α-subcompact, then κ is β-subcompact.

How strong is subcompactness?

Following an old argument of Magidor, we get:

Proposition

1. *If κ is $2^{<\alpha}$-supercompact, then κ is α-subcompact.*
2. *If κ is $(2^{(\lambda^{<\kappa})})^+$-subcompact, then κ is λ-supercompact.*

In particular, κ is supercompact if and only if κ is α-subcompact for every $\alpha > \kappa$.

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Since $\pi``\bar{\alpha}^+$ is $\bar{\kappa}$-closed and unbounded in λ, D is also unbounded in λ; in particular, $|D| \geq \bar{\alpha}^+$.
For each $\delta \in D$, C_δ is an initial segment of C_λ, which itself has order type at most α (by the definition of square).
Thus, $\{\mathrm{ot}(C_\delta) \mid \delta \in D\}$ is a set of at least $\bar{\alpha}^+$-many distinct ordinals less that $\alpha = \pi(\bar{\alpha})$ in the image of π. $\lightning$ $\square$

Optimality

Assuming GCH, the previous result is in some sense optimal:

Theorem (under GCH)

Let

$$I = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \textit{ is } \alpha^+\textit{-subcompact})\}.$$

Then there is a cofinality-preserving class forcing $\mathbb{P}$ such that for any $\mathbb{P}$-generic G the following hold.

1. *If $\kappa < \alpha$ are such that $V \vDash \kappa$ is α-subcompact, then*

   $$V[G] \vDash \kappa \textit{ is } \alpha\textit{-subcompact}.$$

   In particular, $I^{V[G]} = I$.
2. *$\square_\alpha$ holds in $V[G]$ for all $\alpha \notin I$.*

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- at other stages, does nothing (is the trivial forcing).

Because the forcing is trivial from κ to α for α-subcompact κ, the embeddings witnessing α-subcompactness lift automatically, to

$$\begin{aligned}\pi' &: (H_{\bar{\alpha}}^{V[G]}, \in, \bar{\sigma}_G) \to (H_\alpha^{V[G]}, \in, \sigma_G)\\ &: \tau_G \mapsto (\pi(\tau))_G.\end{aligned}$$

This is elementary because if $p \Vdash \varphi(\tau)$, then $\pi(p) \Vdash \varphi(\pi(\tau))$, and the forcing is trivial everywhere on the relevant part of $\mathbb{P}$ where π is not the identity.

What about other large cardinals?

Maybe other large cardinals have some impact too, that we've overlooked. If this forcing destroyed other large cardinals, then perhaps that would indicate that our earlier results aren't so optimal after all.

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It seems that this scenario doesn't occur (at least for a *very* large test case):

Definition

A cardinal κ is ω-superstrong (I2 in the notation of Kanamori) if and only if there is an elementary embedding $j : V \to M$ with critical point κ such that, if we let $\lambda = \sup_{n\in\omega}(j^n(\kappa))$, $V_\lambda \subset M$.

Proposition

The forcing iteration $\mathbb{P}$ of the theorem above preserves all ω-superstrong cardinals.

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Proposition

The forcing iteration $\mathbb{P}$ of the theorem above preserves all ω-superstrong cardinals.

Again, the large cardinal is preserved because the forcing is trivial everywhere that counts.

Stationary reflection

- Stationary reflection at α^+ implies the failure of $\square_\alpha$ (as does α^+-subcompactness).
- α^{++}-subcompactness implies stationary reflection at α^+.

Theorem (under GCH)

Let I be as defined above, and similarly let

$$I^+ = \{\alpha \mid \exists\kappa \leq \alpha(\kappa \text{ is } \alpha^{++}\text{-subcompact})\} \subseteq I.$$

Then there is a cofinality-preserving class forcing $\mathbb{P}$ such that for any $\mathbb{P}$-generic G the following hold.

1. *If $\kappa \leq \alpha$ are such that $V \vDash \kappa$ is α-subcompact, then $V[G] \vDash \kappa$ is α-subcompact. In particular, $I^{V[G]} = I$ and $(I^+)^{V[G]} = I^+$.*
2. *Stationary reflection at α^+ fails in $V[G]$ for all $\alpha \notin I^+$.*
3. *$\square_\alpha$ holds in $V[G]$ for all $\alpha \notin I$.*

Moreover, $\mathbb{P}$ preserves all ω-superstrong cardinals.