Ideals with bases of unbounded Borel complexity

Szymon Głąb, Piotr Borodulin–Nadzieja

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$\mathcal{M}(\mathcal{I})$

Let $\mathcal{I}$ be a σ-ideal. $\mathcal{M}(\mathcal{I}) = \{A \in X \times X : \exists B \supset A, B \text{ Borel}, B_x \in \mathcal{I}\}$

$\mathcal{M}_\alpha(\mathcal{I})$ is a σ-ideal generated by $\mathcal{M}(\mathcal{I}) \cap \Sigma^0_\alpha$.

Since $\mathcal{M}(\mathcal{I})$ has a Borel base, then $\mathcal{M}(\mathcal{I}) = \bigcup \mathcal{M}_\alpha(\mathcal{I})$.

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Ideals $\mathcal{M}(\mathcal{F},\mathcal{I})$

Let $\mathcal{F}\subset X^X$. $Y\subset X^2$ belongs to $\mathcal{M}(\mathcal{F},\mathcal{I})$ whenever $Y\in\mathcal{M}(\mathcal{I})$ and Y can be covered by a Borel set $B\subset X^2$ such that $\{x\colon (x,f(x))\in B\}\in\mathcal{I}$ for every $f\in\mathcal{F}$.

A family of functions $\mathcal{F}\subseteq X^X$ is *ubiquitous with respect to an ideal* $\mathcal{I}$ (or $\mathcal{I}$–ubiquitous) if for every Borel function $g\colon X\to X$ there is a Borel set $B\notin\mathcal{I}$ and a function $f\in\mathcal{F}$ such that $f|B=g|B$.

The family of continuous functions is a natural example of $\mathcal{N}ull$– and $\mathcal{M}eager$–ubiquitous family (Luzin Theorem and Nikodym Theorem).

On the other hand, there are families of Borel functions $f\colon[0,1)\to[0,1)$ which are closed under the addition modulo 1 but are not ubiquitous neither with respect to $\mathcal{N}ull$ nor to $\mathcal{M}eager$ ideals: the empty family, the constant functions, the linear functions, polynomials.

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Ideals $\mathcal{M}(\mathcal{F}, \mathcal{I})$

Let X be a Polish group. Let $\mathcal{I}$ be either the σ–ideal of meager subsets of X or a σ–ideal of null subsets of X with respect to a right–invariant σ–finite measure on X.

Theorem

Let $\mathcal{F} \subseteq X^X$ be a family of Borel functions which is not $\mathcal{I}$-ubiquitous. Assume that $\mathcal{F}$ is left shift invariant, i.e. for any $f \in \mathcal{F}$ and $y \in X$ the function $x \mapsto y \cdot f(x)$ belongs to $\mathcal{F}$. Then $\mathcal{M}_{\alpha+2}(\mathcal{F}, \mathcal{I}) \setminus \mathcal{M}_\alpha(\mathcal{I}) \neq \emptyset$ for every $3 \leq \alpha < \omega_1$.

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Theorem (Cichoń, Pawlikowski)

Assume $\mathcal{I}$ is a σ–ideal of subsets of an uncountable Polish space X such that $X\notin\mathcal{I}$. For every $\alpha<\omega_1$ there is a Π^0_α set $A\subseteq X^2$ such that for every $M\in\mathcal{M}_\alpha(\mathcal{I})$ there is $x\in X$ such that $\emptyset\neq A_x\subseteq M^c_x$. If, additionally, $\mathcal{I}$ is Σ^0_α-on-Π^0_α, then we can assume that $A^c_x\in\mathcal{I}$ for every $x\in\pi_1[A]$.

Theorem (Holický)

Suppose X is an uncountable Polish space. Let $\mathcal{I}$ be a σ–ideal of subsets of X which is Σ^0_α-on-Π^0_α for some $2\leq\alpha<\omega_1$ and which contains all singletons. Let $A\subseteq X^2$ be such that $A_x\notin\mathcal{I}$ for every $x\in\pi_1[A]$. If A is of class Σ^0_α, then there is a Σ^0_α–measurable uniformization of A.

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$\mathcal{M}(\mathcal{I})$ can be seen as $\{\emptyset\} \otimes \mathcal{I}$.

$\mathcal{N}ull \otimes \mathcal{N}ull$, $\mathcal{N}ull \otimes \mathcal{M}eager$ etc. have bases of bounded Borel complexity.

property (M)

We will say that an ideal $\mathcal{J}$ of subsets of a Polish space X has *property (M)* if there is a Borel function $f: X \to [0,1]$ such that $f^{-1}[\{x\}] \notin \mathcal{J}$ for every $x \in [0,1]$.

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Let $\mathcal{I}$ be a σ–ideal of subsets of an uncountable Polish space X. Suppose $\mathcal{I}$ has a Borel base, is Σ^0_α-on-Π^0_α for each $\alpha < \omega_1$ and contains all singletons. If a σ–ideal $\mathcal{J}$ of subsets of X has property (M) then there is $\beta < \omega_1$ such that $(\mathcal{J} \otimes \mathcal{I})_\alpha \subsetneq (\mathcal{J} \otimes \mathcal{I})_{\alpha+2}$ for each $\alpha > \beta$.

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Set theoretic properties

Proposition.

If $\mathcal{I}$ is a σ–ideal of subsets of X such that $X\notin\mathcal{I}$ and $\mathcal{F}\subseteq X^X$, then $\mathcal{M}(\mathcal{F},\mathcal{I})$ has property (M). Also, if a σ–ideal $\mathcal{J}$ has property (M), then $\mathcal{J}\otimes\mathcal{I}$ has property (M).

Proposition.

If $\mathcal{I}$ has the complex Borel base property, then $\mathrm{add}(\mathcal{I})=\omega_1$.

Proposition.

If $\mathcal{I}$ is a σ–ideal of subsets of X and $\mathcal{F}\subseteq X^X$, then

(i) $\mathrm{cov}(\mathcal{M}(\mathcal{F},\mathcal{I}))=\mathrm{cov}(\mathcal{I})$ provided $\mathcal{I}$ has a Borel base;

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