Automatic ordinals and linear orders

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Finite automata

Let us fix a finite alphabet Σ. A *finite automaton* A consists of

- a finite set S of states,
- an initial state,
- a transition function $\Delta : S \times \Sigma \to S$, and
- a subset of S of accepting states.

An input word (string) is read from beginning to end and is accepted or rejected according to the end state.

Automatic structures

An automaton A recognizes a set of words if A accepts exactly the words in the set.

Definition

A (word) automatic structure $(M, R_0, ..., R_n)$ *is (isomorphic to) a structure with domain a set of finite words in a finite alphabet. The domain and the relations of the structure are recognized by finite automata.*

Definition

An ordinal γ *is automatic if* $(\gamma, <)$ *is automatic.*

Examples

Examples:

- $(\omega, +, <)$
- $(\mathbb{Q}, <)$
- finitely generated abelian groups

Non-examples:

- $(\omega^\omega, <)$ (Delhommé)
- $(\omega, \times)$
- the random graph (Delhommé)

Motivation

Why study automatic structures and computable structures?

They are often simpler than arbitrary structures (in a given class):

- the $\exists^{\infty}$-theory of any automatic structure is decidable
- automatic linear orders have finite Cantor-Bendixson rank
- the isomorphism problem for automatic ordinals is decidable.

Counterexample:

- there are automatic wellfounded relations of arbitrary large height below ω_1^{CK} (Khoussainov-Minnes).

Motivation

Do structures recognized by automata with infinite running time α have similar properties?

The automatic ordinals are exactly those below ω^ω (Delhommé). Which ordinals are α-automatic?

Results

Proposition (Stephan-S.)

Suppose $\alpha = \omega \cdot \beta = \omega^{\gamma}$. *Then* $\omega^{\beta \cdot \omega}$ *is the supremum of the* α*-automatic ordinals.*

Proposition (Stephan-S.)

Suppose $\alpha = \omega \cdot \beta = \omega^{\gamma}$. *Then* $\beta \cdot \omega$ *is the supremum of ranks of* α*-automatic linear orders.*

Hence the power of α-automata increases with every power of ω.

α-automata

Let us fix a limit ordinal α and an extra symbol $\diamond$. A *finite α-word* is a word of length α with the letter $\diamond$ almost everywhere.

An *α-automaton* is a finite automaton with a limit transition function which maps the set of states appearing cofinally often before a limit to the state at the limit (similar automata have been studied by Büchi, Choueka, Wojciechowski).

A word is accepted or rejected according to the state at time α.

α-automatic structures

Definition

A structure is (finite word) α-automatic if it is (isomorphic to) a structure whose domain consist of finite α-words and the domain and relations are recognized by α-automata.

Properties:

The $\exists^{\infty}$-theory of any α-automatic structure is decidable and every ω^{γ}-automatic presentation restricted to ω^{ω} represents an elementary substructure.

The class of α-automatic structures is closed under finite products.

Every $(\alpha \cdot n)$-automatic structure is α-automatic.

α-automatic ordinals

Example

Let $(n_0, ..., n_k) <^* (m_0, ..., m_l)$ if

- $k = l$ and $n_i < m_i$ for $i \leqslant n$ least with $n_i \neq m_i$, or
- $k < l$.

This is a wellorder on $\omega^{<\omega}$ of order type ω^ω. Let us represent $(n_0, ..., n_k)$ by the (finite) ω^2-word $0^{n_0}1 \diamond^\omega 0^{n_1}1 \diamond^\omega ...0^{n_k}1\diamond^{\omega^2}$.

Hence ω^ω is ω^2-automatic. Similarly ω^β is α-automatic, where $\alpha = \omega \cdot \beta$.

Automatic product

Suppose C, D are sets of ordinals. Let $tp(C)$ denote the order type of C. Let $tp(C, D)$ denote the isomorphism type of $(C \cup D, C, D, <)$.

Definition

$\alpha *_{aut} \beta$ is the supremum of the ordinals γ such that there is $(C_\delta : \delta < \epsilon)$ with $\gamma = \bigcup_{\delta<\epsilon} C_\delta$ and

- $\forall \delta < \epsilon \; tp(C_\delta) \leqslant \alpha$,
- *there are only finitely many $tp(C_\delta, C_\eta)$ for $\delta, \eta < \epsilon$, and*
- *let for $\mu < \alpha$ $Tr_\mu = \{C_\delta(\mu) : \delta < \epsilon\}$ (the trace of μ). Then $\forall \mu < \alpha \; tp(Tr_\mu) \leqslant \beta$.*

Commutative product

Definition (Hessenberg)

Suppose $\alpha = \Sigma_{i<m}\omega^{\alpha_i}$ *and* $\beta = \Sigma_{j<n}\omega^{\beta_j}$ *are in Cantor normal form. The commutative sum* $\alpha \oplus \beta$ *is the sum of all* ω^{α_i} *and* ω^{β_j} *arranged in decreasing order. The commutative product* $\alpha \otimes \beta$ *is the sum of all* $\omega^{\alpha_i \oplus \beta_j}$ *arranged in decreasing order.*

Commutative sum and product are strictly monotone in both coordinates.

Automatic product

Lemma

$\alpha *_{aut} \beta = \alpha \otimes \beta$ for all α, β.

Proof sketch:

Suppose $\alpha = \omega^{\omega^{\alpha'}}$ and $\beta = \omega^{\omega^{\beta'}}$.

Case $\alpha \otimes \beta = \alpha \cdot \beta$: Let C_γ be sets of ordinals of order type α with

- sup $C_\gamma <$ min C_δ for all $\gamma < \delta < \beta$.

Case $\alpha \otimes \beta = \beta \cdot \alpha$: Same with

- $C_\gamma(\mu) < C_\delta(\nu)$ for all $\mu < \nu < \alpha$ and $\gamma, \delta < \beta$ and
- $C_\gamma(\mu) < C_\delta(\mu)$ for all $\mu < \alpha$ and $\gamma < \delta < \beta$.

Automatic product

Suppose $(C_\delta : \delta < \epsilon)$ as in definition of $\alpha *_{aut} \beta$ and $\alpha = \omega^{\bar{\alpha}}$ and $\beta = \omega^{\bar{\beta}}$.

Assume $tp(C_\gamma) = tp(C_\delta)$ for all $\gamma < \delta < \epsilon$ and

- $\sup C_\gamma = \sup C_\delta$ (case 1) or
- $\sup C_\gamma < \min C_\delta$ (case 2) for all $\gamma < \delta < \epsilon$,

otherwise split.

Case 1: Every proper inital segment is bounded by $\alpha' *_{aut} \beta = \alpha' \otimes \beta < \alpha \otimes \beta$ for some $\alpha' < \alpha = tp(C_\gamma)$.

Case 2: ... bounded by $\alpha *_{aut} \beta' = \alpha \otimes \beta'$ for some $\beta' < \beta$. □

α-automatic ordinals

Proposition

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\omega^{\beta \cdot \omega}$ is the supremum of the α-automatic ordinals.

Proof sketch:

Suppose there is an α-automatic structure of order type at least $\omega^{\beta \cdot \omega}$. Let $u \downarrow$ denote the set of predecessors of u. We pick an element u_n with $tp(u_n \downarrow) = \omega^{\beta \cdot n}$ for each $n \geqslant 1$. Write

$$u_n \downarrow = X_{u_n} \sqcup \bigsqcup_{|v|=|u_n|} Y_v^{u_n}$$

where $X_{u_n} = \{x : |x| < |u_n| \ \& \ x < u_n\}$ and $Y_v^u = \{vw : vw < u\}$.

α-automatic ordinals

$X_{u_n} = \{x : |x| < |u_n| \;\&\; x < u_n\}$
$Y^u_v = \{vw : vw < u\}$

Then $u_n \downarrow -X_{u_n}$ is an automatic product of the sets $Y^{u_n}_v$.

- $tp(X_{u_n}) < \omega^\beta$ since X_{u_n} is $|u_n|$-automatic and $|u_n| < \alpha$
- $tp(Tr_\delta) < \omega^\beta$ since Tr_δ is α'-automatic for some $\alpha' < \alpha$.

Hence there are words v_n with $tp(Y^{u_n}_{v_n}) = \omega^{\beta \cdot n}$ for each $n \geqslant 1$. But the number of types is bounded by the number of states. Contradiction. $\square$

α-automatic linear orders

Definition

A linear order C is an automatic product of linear orders A and B if there are sequences $(C_\gamma : \gamma < \epsilon)$ of subsets of C and $(f_\gamma : \alpha \to C_\gamma : \gamma < \epsilon)$ of onto functions with

- $C = \bigcup_{\gamma<\epsilon} C_\gamma$
- $C_\gamma \hookrightarrow A$ *for all* $\gamma < \epsilon$
- *for each n, there are only finitely many $tp(f_{\gamma_0}, ..., f_{\gamma_n})$ for $\gamma_i < \epsilon$.*
- *let for $\mu < \alpha$ $g_\mu(\gamma) = f_\gamma(\mu)$. Then $\forall \mu < \alpha$ $ran(g_\mu) \hookrightarrow B$.*

α-automatic linear orders

Definition

Suppose $(C, \leqslant)$ is a linear order. Let rk$(\leqslant) = 0$ if the domain is finite. Let rk$(\leqslant) \leqslant \alpha$ if it is a finite sum of $\mathbb{Z}$-sums of linear orders of rank $< \alpha$.

A linear order L is *scattered* if $(\mathbb{Q}, <) \not\hookrightarrow L$. Every scattered linear order has a rank.

α-automatic linear orders

Lemma

Suppose a scattered linear order C is an automatic product of A and B. Then $rk(C) \leqslant rk(A) \oplus rk(B)$.

Proposition

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\beta \cdot \omega$ is the supremum of ranks of α-automatic linear orders.

Questions

What is the supremum of the heights of α-automatic wellfounded partial orders?

Can every countable infinite word α-automatic structure be represented by a finite word α-automatic structure?

Is the random graph α-automatic?

Is the isomorphism problem for automatic linear orders decidable?