Automatic ordinals and linear orders

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Winter school in set theory Hejnice, 3 February 2011
Let us fix a finite alphabet $\Sigma$. A \textit{finite automaton} $A$ consists of

- a finite set $S$ of states,
- an initial state,
- a transition function $\Delta : S \times \Sigma \to S$, and
- a subset of $S$ of accepting states.

An input word (string) is read from beginning to end and is accepted or rejected according to the end state.
Automatic structures

An automaton $A$ recognizes a set of words if $A$ accepts exactly the words in the set.

**Definition**

A (word) automatic structure $(M, R_0, ..., R_n)$ is (isomorphic to) a structure with domain a set of finite words in a finite alphabet. The domain and the relations of the structure are recognized by finite automata.

**Definition**

An ordinal $\gamma$ is automatic if $(\gamma, <)$ is automatic.
Examples

Examples:
- \((\omega, +, <)\)
- \((\mathbb{Q}, <)\)
- finitely generated abelian groups

Non-examples:
- \((\omega^\omega, <)\) (Delhommé)
- \((\omega, \times)\)
- the random graph (Delhommé)
Motivation

Why study automatic structures and computable structures?

They are often simpler than arbitrary structures (in a given class):
- the $\exists^\omega$-theory of any automatic structure is decidable
- automatic linear orders have finite Cantor-Bendixson rank
- the isomorphism problem for automatic ordinals is decidable.

Counterexample:
- there are automatic wellfounded relations of arbitrary large height below $\omega_1^{CK}$ (Khoussainov-Minnes).
Motivation

Do structures recognized by automata with infinite running time $\alpha$ have similar properties?

The automatic ordinals are exactly those below $\omega^\omega$ (Delhommé). Which ordinals are $\alpha$-automatic?
Results

Proposition (Stephan-S.)

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\omega^{\beta \cdot \omega}$ is the supremum of the $\alpha$-automatic ordinals.

Proposition (Stephan-S.)

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\beta \cdot \omega$ is the supremum of ranks of $\alpha$-automatic linear orders.

Hence the power of $\alpha$-automata increases with every power of $\omega$. 
$\alpha$-automata

Let us fix a limit ordinal $\alpha$ and an extra symbol $\diamond$. A finite $\alpha$-word is a word of length $\alpha$ with the letter $\diamond$ almost everywhere.

An $\alpha$-automaton is a finite automaton with a limit transition function which maps the set of states appearing cofinally often before a limit to the state at the limit (similar automata have been studied by Büchi, Choueka, Wojciechowski).

A word is accepted or rejected according to the state at time $\alpha$. 
α-automatic structures

Definition

A structure is (finite word) α-automatic if it is (isomorphic to) a structure whose domain consist of finite α-words and the domain and relations are recognized by α-automata.

Properties:

The $\exists^\infty$-theory of any α-automatic structure is decidable and every $\omega^\gamma$-automatic presentation restricted to $\omega^\omega$ represents an elementary substructure.

The class of α-automatic structures is closed under finite products.

Every $(\alpha \cdot n)$-automatic structure is α-automatic.
**Example**

Let \((n_0, \ldots, n_k) \prec (m_0, \ldots, m_l)\) if

- \(k = l\) and \(n_i < m_i\) for \(i \leq n\) least with \(n_i \neq m_i\), or
- \(k < l\).

This is a wellorder on \(\omega^{<\omega}\) of order type \(\omega^\omega\). Let us represent \((n_0, \ldots, n_k)\) by the (finite) \(\omega^2\)-word 0\(^{n_0}\)1 \(\diamond\omega\) 0\(^{n_1}\)1 \(\diamond\omega\) \ldots 0\(^{n_k}\)1 \(\diamond\omega^2\).

Hence \(\omega^\omega\) is \(\omega^2\)-automatic. Similarly \(\omega^\beta\) is \(\alpha\)-automatic, where \(\alpha = \omega \cdot \beta\).
Automatic product

Suppose $C, D$ are sets of ordinals. Let $tp(C)$ denote the order type of $C$. Let $tp(C, D)$ denote the isomorphism type of $(C \cup D, C, D, \prec)$.

**Definition**

$\alpha *_{\text{aut}} \beta$ is the supremum of the ordinals $\gamma$ such that there is $(C_\delta : \delta < \epsilon)$ with $\gamma = \bigcup_{\delta < \epsilon} C_\delta$ and

- $\forall \delta < \epsilon \; tp(C_\delta) \leq \alpha$,
- there are only finitely many $tp(C_\delta, C_\eta)$ for $\delta, \eta < \epsilon$, and
- let for $\mu < \alpha \; Tr_\mu = \{C_\delta(\mu) : \delta < \epsilon\}$ (the trace of $\mu$). Then $\forall \mu < \alpha \; tp(Tr_\mu) \leq \beta$. 
Commutative product

Definition (Hessenberg)

Suppose $\alpha = \sum_{i<m} \omega^{\alpha_i}$ and $\beta = \sum_{j<n} \omega^{\beta_j}$ are in Cantor normal form. The commutative sum $\alpha \oplus \beta$ is the sum of all $\omega^{\alpha_i}$ and $\omega^{\beta_j}$ arranged in decreasing order. The commutative product $\alpha \otimes \beta$ is the sum of all $\omega^{\alpha_i \oplus \beta_j}$ arranged in decreasing order.

Commutative sum and product are strictly monotone in both coordinates.
Automatic product

**Lemma**

$\alpha \ast_{\text{aut}} \beta = \alpha \otimes \beta$ for all $\alpha, \beta$.

**Proof sketch:**

Suppose $\alpha = \omega^{\omega^{\alpha'}}$ and $\beta = \omega^{\omega^{\beta'}}$.

Case $\alpha \otimes \beta = \alpha \cdot \beta$: Let $C_\gamma$ be sets of ordinals of order type $\alpha$ with

- $\sup C_\gamma < \min C_\delta$ for all $\gamma < \delta < \beta$.

Case $\alpha \otimes \beta = \beta \cdot \alpha$: Same with

- $C_\gamma(\mu) < C_\delta(\nu)$ for all $\mu < \nu < \alpha$ and $\gamma, \delta < \beta$ and
- $C_\gamma(\mu) < C_\delta(\mu)$ for all $\mu < \alpha$ and $\gamma < \delta < \beta$. 

Automatic product

Suppose \((C_\delta : \delta < \epsilon)\) as in definition of \(\alpha \ast_{aut} \beta\) and \(\alpha = \omega^{\bar{\alpha}}\) and \(\beta = \omega^{\bar{\beta}}\).

Assume \(tp(C_\gamma) = tp(C_\delta)\) for all \(\gamma < \delta < \epsilon\) and

- \(\sup C_\gamma = \sup C_\delta\) (case 1) or
- \(\sup C_\gamma < \min C_\delta\) (case 2) for all \(\gamma < \delta < \epsilon\),

otherwise split.

Case 1: Every proper initial segment is bounded by \(\alpha' \ast_{aut} \beta = \alpha' \otimes \beta < \alpha \otimes \beta\) for some \(\alpha' < \alpha = tp(C_\gamma)\).

Case 2: ... bounded by \(\alpha \ast_{aut} \beta' = \alpha \otimes \beta'\) for some \(\beta' < \beta\). \(\square\)
**α-automatic ordinals**

**Proposition**

Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\omega^{\beta \cdot \omega}$ is the supremum of the $\alpha$-automatic ordinals.

**Proof sketch:**

Suppose there is an $\alpha$-automatic structure of order type at least $\omega^{\beta \cdot \omega}$. Let $u \downarrow$ denote the set of predecessors of $u$. We pick an element $u_n$ with $tp(u_n \downarrow) = \omega^{\beta \cdot n}$ for each $n \geq 1$. Write

$$u_n \downarrow = X_{u_n} \sqcup \bigsqcup_{|v|=|u_n|} Y_{v}^{u_n}$$

where $X_{u_n} = \{x : |x| < |u_n| \& x < u_n\}$ and $Y_{v}^{u} = \{vw : vw < u\}$. 
$\alpha$-automatic ordinals

$X_{u_n} = \{x : |x| < |u_n| \& x < u_n\}$
$Y^u_v = \{vw : vw < u\}$

Then $u_n \downarrow -X_{u_n}$ is an automatic product of the sets $Y^u_v$.

- $tp(X_{u_n}) < \omega^\beta$ since $X_{u_n}$ is $|u_n|$-automatic and $|u_n| < \alpha$
- $tp(Tr_\delta) < \omega^\beta$ since $Tr_\delta$ is $\alpha'$-automatic for some $\alpha' < \alpha$.

Hence there are words $v_n$ with $tp(Y^u_{v_n}) = \omega^{\beta \cdot n}$ for each $n \geq 1$. But the number of types is bounded by the number of states.

Contradiction. \qed
\( \alpha \)-automatic linear orders

**Definition**

A linear order \( C \) is an automatic product of linear orders \( A \) and \( B \) if there are sequences \(( C_\gamma : \gamma < \epsilon )\) of subsets of \( C \) and \(( f_\gamma : \alpha \rightarrow C_\gamma : \gamma < \epsilon )\) of onto functions with

- \( C = \bigcup_{\gamma < \epsilon} C_\gamma \)
- \( C_\gamma \hookrightarrow A \) for all \( \gamma < \epsilon \)
- for each \( n \), there are only finitely many \( \text{tp}(f_{\gamma_0}, ..., f_{\gamma_n}) \) for \( \gamma_i < \epsilon \).
- let for \( \mu < \alpha \) \( g_\mu(\gamma) = f_\gamma(\mu) \). Then \( \forall \mu < \alpha \ \text{ran}(g_\mu) \hookrightarrow B \).
**α-automatic linear orders**

### Definition

Suppose $(C, \leq)$ is a linear order. Let $rk(\leq) = 0$ if the domain is finite. Let $rk(\leq) \leq \alpha$ if it is a finite sum of $\mathbb{Z}$-sums of linear orders of rank $< \alpha$.

A linear order $L$ is scattered if $(\mathbb{Q}, <) \leftrightarrow L$. Every scattered linear order has a rank.
Lemma
Suppose a scattered linear order $C$ is an automatic product of $A$ and $B$. Then $rk(C) \leq rk(A) \oplus rk(B)$.

Proposition
Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\beta \cdot \omega$ is the supremum of ranks of $\alpha$-automatic linear orders.
Questions

What is the supremum of the heights of $\alpha$-automatic wellfounded partial orders?

Can every countable infinite word $\alpha$-automatic structure be represented by a finite word $\alpha$-automatic structure?

Is the random graph $\alpha$-automatic?

Is the isomorphism problem for automatic linear orders decidable?