Pcf theory and cardinal invariants of the reals

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Winter School in Abstract Analysis
section Set Theory

The beginning

- Is every set definable?
- Is every set definable in some generic extension?
- Hamkin's question: **Is every set X definable in some cardinal preserving extension of the ground model?**
- What about $X \subset \omega$?
- Cardinal exponentiation can not help: it may collapse cardinals
- Use the spectrum of some cardinal invariant to code!

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Coding by MADness

- $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega$ is MAD$\}$
- spectrum($\mathfrak{a}$)$= \{|\mathcal{A}| : \mathcal{A} \subset [\omega]^\omega$ **is MAD**$\}$
- Let $X \subset \omega$.
  If CH holds then there is a c.c.c poset P such that

$$V^P \models \text{“} n \in X \text{ iff } \aleph_{n+1} \in \text{spectrum}(\mathfrak{a})\text{”}.$$

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If CH holds then for each $X \subset \omega$ there is a c.c.c poset P such that

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- No problem with $\{\aleph_1, \aleph_2, \dots\}$ and $\{\aleph_{\omega+2}, \aleph_{\omega+3}, \dots\}$
- **find a c.c.c poset P s.t.** $V^P \models \{\aleph_n : 1 \le n < \omega\} \subset \text{spectrum}(\mathfrak{a}) \wedge \aleph_{\omega+1} \notin \text{spectrum}(\mathfrak{a})$

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Spectrum of a cardinal invariant

Characterize spectrum $(\mathfrak{x})$ for different cardinal invariants!

- $\mathfrak{x}$ cardinal invariant (e.g. $\mathfrak{a}$, $\mathfrak{b}$)
- $\mathfrak{x} = \min\{|X| : X \in \mathfrak{X}_{\mathfrak{x}}\}$ or $\mathfrak{x} = \sup\{|X| : X \in \mathfrak{X}_{\mathfrak{x}}\}$
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Cofinality spectrum of groups

- $S(\omega)$ the group of all permutations of the natural numbers
- Define the **cofinality spectrum** of $S(\omega)$ as follows:
- $\lambda \in \mathrm{CF}(S(\omega))$ iff $S(\omega)$ is the union of an increasing chain of λ proper subgroups.
- Shelah and Thomas: (1) if $\{\kappa_n : n < \omega\} \in \left[\mathrm{CF}(S(\omega))\right]^{\omega}$ increasing then $pcf(\{\kappa_n : n < \omega\}) \subset \mathrm{CF}(S(\omega))$
  (2) IF GCH holds and $K \subset Reg$ s.t (i), (ii) and (iii) hold,
  THEN $CF(S(\omega)) = K$ in some c.c.c generic extension
- **Problem**: Full characterization of $CF(S(\omega))$

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The spectrum of $\mathfrak{b}$

Shelah-Thomas: $CF(S(\omega))$ is pcf-closed.

- What about the spectrum of $\mathfrak{b}$?
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- $\mathfrak{b}$ is the minimal size of an unbounded chain in $\langle \omega^\omega, \leq^* \rangle$
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Theorem (Farah)

Assume GCH. Given any set K of uncountable regular cardinals there is a c.c.c poset $\mathcal{H}_K$ s. t.
$V^{\mathcal{H}_K} \models chain - spectrum(\mathfrak{b}) = K \cup \{\aleph_1\}$.

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Additivity spectrum of ideals

chain $-$ *spectrum*$(\mathfrak{b})$ can be "arbitrary"

- $\mathcal{B} = \{B \subset \omega^\omega : B$ is $\le^*$-bounded in $\langle \omega^\omega, \le^* \rangle\}$
- $\Phi(b) = \{x \in \omega^\omega : x \le^* b\}$
- Φ embeds $\langle \omega^\omega, \le^* \rangle$ into $\langle \mathcal{B}, \subset \rangle$
- $\mathcal{B}$ the σ-ideal generated by the compact subsets of the irrationals.
- $\mathfrak{b} = add(\mathcal{B})$
- If $\mathcal{I}$ is an ideal, let $\mathrm{ADD}(\mathcal{I})$ be **the additivity spectrum of $\mathcal{I}$**:
  $\kappa \in \mathrm{ADD}(\mathcal{I})$ iff
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chain – *spectrum*($\mathfrak{b}$) can be "arbitrary"

- $\mathcal{B} = \{B \subset \omega^\omega : B$ is $\le^*$-bounded in $\langle \omega^\omega, \le^* \rangle\}$
- $\Phi(b) = \{x \in \omega^\omega : x \le^* b\}$
- Φ embeds $\langle \omega^\omega, \le^* \rangle$ into $\langle \mathcal{B}, \subset \rangle$
- $\mathcal{B}$ the σ-ideal generated by the compact subsets of the irrationals.
- $\mathfrak{b} = add(\mathcal{B})$
- If $\mathcal{I}$ is an ideal, let ADD($\mathcal{I}$) be **the additivity spectrum of $\mathcal{I}$**:
  $\kappa \in$ ADD($\mathcal{I}$) iff
  there is an increasing chain $\{C_\alpha : \alpha < \kappa\} \subset \mathcal{I}$ with $\cup_{\alpha<\kappa} C_\alpha \notin \mathcal{I}$.
- ADD($\mathcal{B}$) $\supseteq$ *chain* – *spectrum*($\mathfrak{b}$).

Additivity spectrum of ideals

$\mathrm{ADD}(\mathcal{I}) = \{\kappa \in \mathfrak{Reg} : \exists \{C_\alpha : \alpha < \kappa\} \nearrow \mathcal{I}$ with $\cup_{\alpha<\kappa} C_\alpha \notin \mathcal{I}.$

- For $C \in \mathcal{I}^+$, let
  $\mathrm{ADD}(\mathcal{I}, C) = \{\kappa \in \mathfrak{Reg} :$
  $\exists$ **increasing** $\{C_\alpha : \alpha < \kappa\} \subset \mathcal{I}$ **s.t.** $\cup_{\alpha<\kappa} C_\alpha = C\}$
- $\mathrm{ADD}(\mathcal{I}) = \cup\{\mathrm{ADD}(\mathcal{I}, C) : C \in \mathcal{I}^+\}.$

Theorem

Assume that $\mathcal{I} \subset \mathcal{P}(I)$ is a σ-complete ideal, $C \in \mathcal{I}^+$, and $A \subset \mathrm{ADD}(\mathcal{I}, C)$ is countable. Then $pcf(A) \subset \mathrm{ADD}(\mathcal{I}, C)$.

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Assume that $\mathcal{I} \subset \mathcal{P}(I)$ is a σ-complete ideal, $C \in \mathcal{I}^+$, and $A \subset \mathrm{ADD}(\mathcal{I}, C)$ is countable. Then $pcf(A) \subset \mathrm{ADD}(\mathcal{I}, C)$.

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$\mathcal{I}$ has the **Hechler property** iff given any σ-directed poset Q there is a c.c.c poset P such that
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Assume that $\mathcal{I} = \mathcal{B}$ or $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} = \mathcal{N}$. Given a nonempty, countable subset A of uncountable regular cardinals, T.F.A.E

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Farah: GCH $\Longrightarrow$ $chain - spectrum(\mathfrak{b})$ is arbitrary (mod $\aleph_1$)

$K = \mathrm{ADD}(\mathcal{B})$ iff $K = pcf(K)$

- Can we prove (some form of) a **strong Hechler theorem**?
  If Q is a **σ-directed poset** then there is a c.c.c poset P s. t.
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- Soukup, L: **Pcf theory and cardinal invariants of the reals**, arXiv:1006.1808v1, to appear in CMUC.