Mathias-Prikry type forcing and dominating real

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Mathias-Prikry type and Laver-Prikry type forcings

Definition

Let $\mathcal{I}$ be an ideal on ω.

Mathias-Prikry type forcing

$$\langle s, F\rangle \in \mathbb{M}_{\mathcal{I}^*} \text{ if } s \in [\omega]^{<\omega} \wedge F \in \mathcal{I}^* \wedge s \cap F = \emptyset$$

ordered by

$$\langle s, F\rangle \leq \langle t, G\rangle \text{ if } s \supset t \wedge F \subset G \wedge s \setminus t \subset G.$$

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Laver-Prikry type forcing

$$T \in \mathbb{L}_{\mathcal{I}^*} \text{ if } T \subset \omega^{\omega} \text{ is tree } \wedge \exists s \in T(\forall t \in T(s \subset t \vee t \subset s)$$
$$\text{and } \forall t \in T(s \subset t \rightarrow \mathrm{Succ}_T(t) = \{n \in \omega : t^\frown n \in T\} \in \mathcal{I}^*)),$$

where such $s \in T$ is called *stem* of T, denoted **stem(T)**.
$\mathbb{L}_{\mathcal{I}^*}$ is ordered by inclusion.

Mathias forcing and $\mathbb{L}_{\mathcal{F}}$ add a dominating real. It is depend on filter $\mathcal{F}$ whether $\mathbb{M}_{\mathcal{F}}$ adds a dominating real.

Theorem (Canjar)

1. *If $\mathcal{U}$ is either rapid ultrafilter or not a P-point ultrafilter, then $\mathbb{M}_{\mathcal{U}}$ adds a dominating real.*
2. *If* CH holds, there exists an ultrafilter $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$ doesn't add a dominating real.

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Question

When does $\mathbb{M}_{\mathcal{I}^*}$ add dominating real?

Decision property

Laver forcing $\mathbb{L}$ and Mathias forcing have decision property.

Theorem

1. *For every sentence ϕ of forcing language, for every $T \in \mathbb{L}$ there exists $S \leq T$ with $\mathbf{stem}(S) = \mathbf{stem}(T)$ such that*

$$S \Vdash_{\mathbb{L}} \phi \text{ or } S \Vdash_{\mathbb{L}} \neg\phi.$$

2. *For every sentence ϕ of forcing language, for every $\langle s, A\rangle \in \mathbb{M}$, there exists infinite $B \subset A$ such that*

$$\langle s, B\rangle \Vdash_{\mathbb{M}} \phi \text{ or } \langle s, B\rangle \Vdash_{\mathbb{M}} \neg\phi.$$

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The decision property doesn't hold for Mathias-Prikry and Laver-Prikry type forcing in general.

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When we use $\mathbb{L}_{\mathcal{I}^*}$, rank argument is important. But we can't define rank for $\mathbb{M}_{\mathcal{I}^*}$ in general. When we use $\mathbb{M}_{\mathcal{I}^*}$, $\mathcal{I}^{<\omega}$ is significant.

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For an ideal $\mathcal{I}$ on ω,

$$\mathcal{I}^{<\omega} = \{A \subset [\omega]^{<\omega} \setminus \{\emptyset\} : \exists I \in \mathcal{I} \forall a \in A (a \cap I \neq \emptyset)\}.$$

Then $\mathcal{I}^{<\omega}$ is an ideal on $[\omega]^{<\omega} \setminus \{\emptyset\}$.

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Theorem

For every sentence ϕ of forcing language, for $s \in [\omega]^{<\omega}$ define

$$X_s = \{t \in [\omega \setminus s]^{<\omega} : \exists F \in \mathcal{I}^* ((s \cup t) \cap F = \emptyset \wedge \langle s \cup t, F \rangle \Vdash \phi)\}$$

Then if $X_s \in (\mathcal{I}^{<\omega})^+$, for every $F \in \mathcal{I}^*$ with $s \cap F = \emptyset$, there exists $\langle s \cup t, G \rangle \leq \langle s, F \rangle$ such that $\langle s \cup t, G \rangle \Vdash_{\mathbb{M}_{\mathcal{I}^*}} \phi$.
If $X_s \in \mathcal{I}^{<\omega}$, for every $F \in \mathcal{I}^*$ with $s \cap F = \emptyset$, there exists $\langle s \cup t, G \rangle \leq \langle s, F \rangle$ such that $\langle s \cup t, G \rangle \Vdash_{\mathbb{M}_{\mathcal{I}^*}} \neg\phi$.

$\mathbb{M}_{\mathcal{I}^*}$ and $\mathcal{I}^{<\omega}$-positive set

Proof.

1. Suppose $X_s \in (\mathcal{I}^{<\omega})^+$. Let $F \in \mathcal{I}^*$ with $s \cap F = \emptyset$. Then $[F]^{<\omega} \cap X_s \neq \emptyset$. Let $t \in X_s \cap [F]^{<\omega}$. By definition of X_s, there exists $H \in \mathcal{I}^*$ such that $\langle s \cup t, H \rangle \Vdash \phi$.

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   Since $\mathcal{I}^*$ is filter, $G = F \cap H \in \mathcal{I}^*$. Since $t \subset F$, $\langle s \cup t, G\rangle \leq \langle s, F\rangle$. So $\langle s \cup t, G\rangle \leq \langle s, F\rangle$ and $\langle s \cup t, G\rangle \Vdash \phi$.

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2. Suppose $X_s \in \mathcal{I}^{<\omega}$.

$\mathbb{M}_{\mathcal{I}^*}$ and $\mathcal{I}^{<\omega}$-positive set

Proof.

1. Suppose $X_s \in (\mathcal{I}^{<\omega})^+$. Let $F \in \mathcal{I}^*$ with $s \cap F = \emptyset$. Then $[F]^{<\omega} \cap X_s \neq \emptyset$. Let $t \in X_s \cap [F]^{<\omega}$. By definition of X_s, there exists $H \in \mathcal{I}^*$ such that $\langle s \cup t, H \rangle \Vdash \phi$.
   Since $\mathcal{I}^*$ is filter, $G = F \cap H \in \mathcal{I}^*$. Since $t \subset F$, $\langle s \cup t, G \rangle \leq \langle s, F \rangle$. So $\langle s \cup t, G \rangle \leq \langle s, F \rangle$ and $\langle s \cup t, G \rangle \Vdash \phi$.
2. Suppose $X_s \in \mathcal{I}^{<\omega}$.
   Let $I \in \mathcal{I}$ such that $\forall x \in X_s (x \cap I \neq \emptyset)$. Let $\langle s, F \rangle \in \mathbb{M}_{\mathcal{I}^*}$. Let $H = F \setminus I \in \mathcal{I}^*$. Then there exists $\langle s \cup t, G \rangle \leq \langle s, H \rangle$ such that $\langle s \cup t, G \rangle$ decide ϕ. Since $t \cap I = \emptyset$, $t \notin X_s$. Therefore $\langle s \cup t, G \rangle \Vdash \neg\phi$.

□

$\mathbb{M}_{\mathcal{I}^*}$ and dominating real

Theorem (Hrušák, Minami)

The following are equivalent.

1. $\mathbb{M}_{\mathcal{I}^*}$ *adds a dominating real.*
2. $\mathcal{I}^{<\omega}$ *is not* $\boldsymbol{P}^+$ *ideal.*

Definition

$\mathcal{J}$ is $\boldsymbol{P}^+$-ideal if for every decreasing sequence $\{X_n : n \in \omega\}$ of $\mathcal{J}$-positive set, there exists $X \in \mathcal{J}^+$ such that $X \subset^* X_n$.

Theorem (Hrušák, Minami)

The following are equivalent.

1. $\mathbb{M}_{\mathcal{I}^*}$ *adds a dominating real.*
2. $\mathcal{I}^{<\omega}$ *is not* $\boldsymbol{P}^+$ *ideal.*

Proof. From (1) to (2).
Let $\dot{g}$ be a $\mathbb{M}_{\mathcal{I}^*}$-name for a dominating real. For $f \in \omega^\omega \cap V$, there exists $s_f \in [\omega]^{<\omega}$, $F_f \in \mathcal{I}^*$ and $n_f \in \omega$ such that

$$\langle s_f, F_f \rangle \Vdash \forall n \geq n_f (f(n) \leq \dot{g}(n)).$$

Since $\mathbf{cf}(\mathfrak{d}) > \omega$, there exists $s \in [\omega]^{<\omega}$ and $n \in \omega$ such that

$$\mathcal{F} = \{f \in \omega^\omega : s_f = s \wedge n_f = n\}$$

is a dominating family. Fix such $s \in [\omega]^{<\omega}$ and $n \in \omega$.
Define

$$X_s = \{t \in [\omega \setminus \max(s)]^{<\omega} : \exists F \in \mathcal{I}^* \exists m \geq n \, (\langle s \cup t, F \rangle \text{ decides } \dot{g}(m))\}.$$

Claim

$$X_s = \{t \in [\omega \setminus \max(s)]^{<\omega} : \exists F \in \mathcal{I}^* \exists m \geq n\,(\langle s \cup t, F\rangle \textit{ decides } \dot{g}(m))\} \in (\mathcal{I}^{<\omega})^+.$$

Let $z_t = \{m \geq n : \exists F \in \mathcal{I}^*(\langle s \cup t, F\rangle \text{ decides } \dot{g}(m))\}$.
Then define $\langle k_t, l_t\rangle \in \omega \times \omega$ for $t \in X_s$ by

$$k_t = \begin{cases} \max(z_t) & \text{if } |z_t| < \omega \\ \min(z_t \setminus \max(t)) & \text{otherwise.} \end{cases}$$

Choose $l_t \in \omega$ so that there exists $F \in \mathcal{I}^*$ so that $\langle s \cup t, F\rangle \Vdash \dot{g}(k_t) = l_t$.
Define $H : X_s \to \omega \times \omega$ by $H(t) = \langle k_t, l_t\rangle$.

Claim

For every $m \in \omega$, $H^{-1}[(\omega \setminus m) \times \omega] \in (\mathcal{I}^{<\omega})^+$.

Let $K = \{k_t : t \in X_s\}$. Let $\{k_i : i \in \omega\}$ be the increasing enumeration of K. Define $L_i = \{l_t : k_i = k_t \wedge t \in X_s\}$.

Claim

$\exists^\infty i \in \omega(|L_i| = \omega)$.

Proof

Assume to the contrary, $\forall^\infty i \in \omega(|L_i| < \aleph_0)$. Then we can define $h : \omega \to \omega$ by

$$h(m) = \begin{cases} \max(L_i) \\ \quad \text{if there exists } i \in \omega \text{ such that } m = k_i \text{ and } |L_i| < \aleph_0. \\ \\ 0 \\ \quad \text{otherwise.} \end{cases}$$

Proof.

Since $\mathcal{F}$ is a dominating family, there exists $f \in \mathcal{F}$ and $m_0 > n$ such that $\forall m \geq m_0 (h(m) \leq f(m))$.

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However there exists $t \in H^{-1}[(\omega \setminus m_0) \times \omega] \cap [F_f]^{<\omega}$ since $H^{-1}[(\omega \setminus m_0) \times \omega] \in (\mathcal{I}^{<\omega})^+$.

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By definition of h, there exists $H \in \mathcal{I}^*$ and $k_t \geq m_0$ such that

$$\langle s \cup t, H \rangle \Vdash \dot{g}(k_t) \leq h(k_t)(\leq f(k_t)).$$

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However $\langle s, F_f \rangle \Vdash \forall m \geq n(f(m) < \dot{g}(m))$ and $\langle s, F_f \rangle$ is compatible with $\langle s \cup t, H \rangle$. It is contradiction.

□

Without loss of generality, we can assume for all $i \in \omega$ $|L_i| = \aleph_0$. Let $Y_m = \{H^{-1}[\bigcup_{m \geq i} L_i]\}$ for $m \geq n$. Then $Y_{m+1} \subset Y_m$.

Claim

$Y_m \in (\mathcal{I}^{<\omega})^+$ for $m \geq n$.

Let $Y \subset^* Y_m$ for $m \geq n$. We shall show $Y \in \mathcal{I}^{<\omega}$.
Assume to the contrary that $Y \in (\mathcal{I}^{<\omega})^+$. Define a function g from ω to ω by

$$g(m) = \begin{cases} \max\{l_t : \exists t \in Y(m = k_t)\} \\ \text{ if there exists } t \in Y \text{ such that } k_t = m \\ \\ 0 \\ \text{ otherwise.} \end{cases}$$

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Since $\mathcal{F}$ is a dominating family, $\exists f \in \mathcal{F}(g \leq^* f)$. Let $m_0 \geq n$ such that $g(m) \leq f(m)$ for $m \geq m_0$.

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Let $Y \subset^* Y_m$ for $m \geq n$. We shall show $Y \in \mathcal{I}^{<\omega}$.
Assume to the contrary that $Y \in (\mathcal{I}^{<\omega})^+$. Define a function g from ω to ω by

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From (2) to (1). Let $\langle X_n : n \in \omega \rangle$ be a decreasing sequence of $(\mathcal{I}^{<\omega})^+$ without pseudointersection in $(\mathcal{I}^{<\omega})^+$. Let $\langle a_k : k \in \omega \rangle$ be an enumeration of $[\omega]^{<\omega} \setminus \{\emptyset\}$. Let $\dot{a}_{gen}$ be a $\mathbb{M}_{\mathcal{I}^*}$-name for $\mathbb{M}_{\mathcal{I}^*}$-generic real($\subset \omega$). Define $\mathbb{M}_{\mathcal{I}^*}$-name $\dot{g}$ for a function from ω to ω by

$$\Vdash \dot{g}(n) = \min\{k : a_k \subset [\dot{a}_{gen}]^{<\omega} \cap X_n \wedge \max(\bigcup\{a_m : l < n \wedge m = \dot{g}(l)\}) < \min(a_k)\}.$$

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We shall show $\dot{g}$ be a dominating real. Let $f \in \omega^\omega \cap V$ and $\langle s, F \rangle \in \mathbb{M}_{\mathcal{I}^*}$. Let

$$I_f = \{a_k \in [\omega]^{<\omega} \setminus \{\emptyset\} : \exists n \in \omega (a_k \in X_n \wedge k \leq f(n))\}.$$

Then $I_f \subset^* X_n$ for every $n \in \omega$. Therefore $I_f \in \mathcal{I}^{<\omega}$ by definition of X_n. Let $\hat{I} \in \mathcal{I}$ such that $\forall a \in I_f (a \cap I \neq \emptyset)$. Then $F \setminus I \in \mathcal{I}^*$ and $[F \setminus I]^{<\omega} \cap I_f = \emptyset$.

Claim

Let $\langle t_n : n < \alpha\rangle$ be a sequence of finite subsets of ω so that

1. $t_n \in [s \cup (F \setminus I)]^{<\omega} \cap X_n$
2. $\max(t_n) < \min(t_{n+1})$
3. $\exists k \in \omega(t_n = a_k \wedge k \leq f(n))$

Then $\alpha \leq |s|$.

Proof of Claim.

If $t \in [F \setminus I]^{<\omega}$, then $t = a_k$ and $t \in X_n$ implies $k > f(n)$ by $[F \setminus I]^{<\omega} \cap I_f = \emptyset$. So by (2), $\alpha \leq |s|$. □

Put $|s|$=m. Then $\langle s, F \setminus I\rangle \leq \langle s, F\rangle$ and

$$\langle s, F \setminus I\rangle \Vdash \forall n > m(f(n) < \dot{g}(n)).$$

ultrafilter case

Definition (Laflamme)

An ultrafilter $\mathcal{U}$ is strong $\boldsymbol{P}$-point if for every ω-sequence of closed subset $\mathcal{C}_n \subset \mathcal{U}$, there exists a partition of ω into finite intervals I_n such that for any sets $B_n \in \mathcal{C}_n$,

$$\bigcup_n (B_n \cap I_n) \in \mathcal{U}.$$

Theorem (Blass-Laflamme)

Suppose $\mathcal{U}$ is an ultrafilter. Then the following are equivalent.

1. *$\mathcal{U}$ is a strong $\boldsymbol{P}$-point.*
2. *$\mathcal{U}^{<\omega}$ is $\boldsymbol{P}^+$ filter.*
3. *$\mathbb{M}_{\mathcal{U}}$ doesn't add a dominating real.*

Thank you!

Reference

1. Michael Hrušák and Hiroaki Minami, "Mathias-Prikry type forcing and Laver-Prikry type forcing", preprint.
2. Andreas Blass, "Strong ***P***-points and the Hrušák-Minami condition", preprint.

Appendix: Ultrafilter

Definition

Let $\mathcal{U}$ be a filter on ω.

1. $\mathcal{U}$ is selective ultrafilter if
   $\forall f \in \omega^\omega \exists U \in \mathcal{U}(f \restriction U$ is one-to-one or constant$)$.
2. $\mathcal{U}$ is nowhere dense ultrafilter if
   $\forall f : \omega \to 2^\omega \exists U \in \mathcal{U}(F[U]$ is nohere dense$)$.
3. $\mathcal{U}$ is rapid if $\forall f \in \omega^\omega \exists U \in \mathcal{U}(\left|U \cap f(n)\right| \leq n)$.
4. $\mathcal{U}$ is P-point ultrafilter if
   $\forall f \in \omega^\omega \exists U \in \mathcal{U}(f \restriction U$ is finite-to-one or constant$)$.

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